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U.T.F. 332 - July 1994

VACUUM CORRELATIONS AT GEODESIC DISTANCE IN QUANTUM GRAVITY

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Abstract

The vacuum correlations of the gravitational field are highly non-trivial to be defined and computed, as soon as their arguments and indices do not belong to a background but become dynamical quantities. Their knowledge is essential however in order to understand some physical properties of quantum gravity, like virtual excitations and the possibility of a continuum limit for lattice theory. In this review the most recent perturbative and non-perturbative advances in this field are presented. (To appear on Riv. Nuovo Cim.)

CONTENTS

1. INTRODUCTION

- 1.1 Vacuum correlations in the absence of gravity ... 3
- 1.2 Gravitational correlations in a physical gauge ... 5
- 1.3 Invariant gravitational correlations ... 6

2. THE MANDELSTAM COVARIANTS

- 2.1 The original definition ... 8
- 2.2 The computation of Tsamis and Woodard ... 9

3. CLASSICAL RADIAL GAUGE AND APPLICATIONS

- 3.1 Main features ... 11
- 3.2 Inversion formulas ... 14
- 3.3 Translation of the origin ... 16
- 3.4 Solutions of (2+1)-gravity ... 18
- 3.5 Motion of test particles in a fluctuating field ... 23

4. RADIAL PROPAGATORS

- 4.1 The gauge projectors method ... 27
- 4.2 Electrodynamics. Regularity properties ... 32
- 4.3 Gravitational propagators ... 36
- 4.4 Relation to the Mandelstam covariant. Significant components ... 39

5. GAUGE-INVARIANT CORRELATIONS

- 5.1 Geometrical definitions ... 42
- 5.2 Dynamics and perturbation scheme ... 44
- 5.3 Two-point functions ... 45
- 5.4 Wilson loops ... 47

6. FURTHER PROPERTIES OF THE WILSON LOOPS

- 6.1 Geometrical and physical definitions ... 50
- 6.2 The dumbbell correlation function ... 52
- 6.3 Geodesic corrections ... 54
- 6.4 The static potential energy ... 57

7. LATTICE GRAVITY

- 7.1 $(R + R^2)$ -gravity ... 59
- 7.2 Regge calculus ... 60
- 7.3 The simulations of Hamber ... 61
- 7.4 The gauge approach and others ... 63

REFERENCES ... 66

1. INTRODUCTION.

The aim of this paper is that of giving an introduction to the issue of vacuum correlations in quantum gravity and a review of the main advances recently made in this field. The whole subject is quite new and in rapid development, so the present account can only be considered as provisional.

From the physical point of view, this is a very interesting field of research, since it concerns the physical modes of propagation of the gravitational field and their quantization, the structure of the vacuum state and the general problem of the observables in quantum gravity.

It is known that a completely consistent theory of quantum gravity has not been established yet. Some of the authors we shall mention in this paper really work in this direction: the simplicial quantum gravity of Hamber (Chapter 7), for instance, aims to put Einstein's gravity on a solid quantum basis by simulating numerically its non-perturbative behaviour; also the original purpose of the work of Tsamis and Woodard (Chapter 2) was that of finding the “true” observables of quantum gravity, free of ultraviolet divergences.

Our philosophy throughout this paper, however, will be mainly that of regarding quantum gravity as an effective quantum field theory, which has General Relativity as its classical limit, but which could go over to some more fundamental theory at very short distances. Our approach will be in some sense “phenomenological”, as we believe that knowledge of the properties of the gravitational vacuum correlations and other observables can help in guiding the fundamental research.

From the technical point of view, the issue is confronted either in a suitable physical gauge (the Mandelstam covariant and the radial gauge, Ch.s 2, 3, 4) and in gauge-invariant form (Ch.s 5, 6, 7). In the following of this Introduction, after giving some general definitions, we shall survey both approaches giving an overall outline of the paper. A more detailed outline for each chapter can be found at the beginning of the chapter itself.

1.1 Vacuum correlations in the absence of gravity.

The vacuum correlation functions are very important quantities in quantum field theory. It was shown by Wightman many years ago [Wightman, 1956] that the knowledge of all the correlation functions of a scalar field $\phi(x)$ gives a complete information about the quantum dynamics of the field. More recently, in the lattice formulations of euclidean quantum field theory, the field correlations characterize the “phases” of the equivalent statistical system. Typically, they behave like $\langle\phi(x)\phi(y)\rangle \sim e^{-|x-y|/\xi}$ outside the region of phase transition, and like $\langle\phi(x)\phi(y)\rangle \sim |x-y|^{-m}$ in correspondence of the transition (where the continuum limit is recovered; see Chapter 7).

In general, for vacuum correlations of a quantum field theory whose fields are denoted by $\Phi_A(x)$, $\Phi_B(x)$... we mean the expectation value on the vacuum state of the product of the fields:

$$G_{AB\dots}(x, y\dots) = \langle \Phi_A(x)\Phi_B(y)\dots \rangle_0 = \frac{1}{z} \int d[\Phi] e^{(i/\hbar)S[\Phi]} \Phi_A(x)\Phi_B(y)\dots \quad (1.1)$$

In this expression the indices A, B, \dots are Lorentz indices (possible “internal” indices are understood) and the coordinates x, y, \dots are the coordinates of Minkowski space.

In the case of free fields – for instance, scalar fields – the function $G(x, y)$ is computed exactly and is called the Pauli-Jordan distribution

$$\langle \phi(x)\phi(y) \rangle_0 = \frac{1}{(2\pi)^3} \int d^4k e^{-ikx} \theta(k^0) \delta(k^2 - m^2).$$

From this expression one can verify that also in a “trivial” system like a free field the vacuum correlations exhibit interesting properties: for instance, they do not vanish when $(x - y)$ is spacelike, although all the observables in x and y commute in this case.

Intuitively, from the statistical point of view, the product of two real random variables vanishes on the average if the two variables are independent, and have thus the same probability to assume sign + or -, with various amplitudes. But since the field Φ propagates, its value in y depends on that in x , and averaging on all the configurations like in (1.1) we find that the correlation increases like $|x - y|^{-m}$ when the two points get closer and closer. Of course, in the functional integral (1.1) all the field configurations are contained, not just the classical configurations which minimize the action; but if Φ oscillates too rapidly at small distances in some configuration, the kinetic term of the action is very big and that configuration is strongly suppressed.

As we saw before, the function G possesses in general, in the absence of gravity, some Lorentz indices. For instance, for the correlation of two vector fields we have

$$G_{\mu\nu}(x, y) = \langle V_\mu(x)V_\nu(y) \rangle_0.$$

This means that the field $V_\mu(x)$ and the field $V_\nu(y)$ can be “rotated” by a Lorentz matrix – the same in x and in y – and the correlation function also transforms correspondingly; for instance, the correlation $\langle V_1(x)V_2(y) \rangle_0$ becomes, in a rotated reference frame, $\langle V'_2(x')V'_3(y') \rangle_0$. What allows us to compare in a consistent way two components of the field in two distinct points is the “rigid” structure of Minkowski space. Also it is obvious – still in the absence of gravity – that the distance $|x - y|$ is a fixed quantity, that does not depend on the field V itself.

If we allow a local invariance in our theory, the situation changes remarkably. This can be seen already in the usual gauge theories, based on “internal” symmetries. In this case, the two fields $A_\mu(x)$ and $A_\nu(y)$ which have to be correlated gauge-transform independently in x and y . This reflects itself in the fact that the correlation function is gauge-dependent. However, the coordinates x and y and the Lorentz indices are still referred to a rigid background.

In the case of gravity, the analogue of the local gauge transformations are the coordinates transformations. There is no fixed background and the only way to give a physical meaning to a correlation function (except for the pure topological content) is that of comparing the fields in two different points by parallel transport. As the distance of the points, the geodesic distance must be considered. But either in the evaluation of the parallel transport and of the geodesic distance the same dynamical field enters, whose correlations we are looking for. It is clear that this will introduce in general new technical difficulties.

1.2 Gravitational correlations in a physical gauge.

If we are interested in all the “components” of the correlation functions, we have to compute them in a suitable preferred physical gauge. A first attempt to the computation of this kind of “physical Green functions” for gravity was done by Tsamis and Woodard (Section 2.2). They computed the vacuum correlations of the Mandelstam covariant (Section 2.1) to one-loop perturbative order, in the hope that these quantities, being in principle observable, would have been free of divergences. Unfortunately, as a result of their hard work (one of the biggest perturbative computations in quantum gravity) they found that the usual divergences are still there, and some new ones appear, which are peculiar of the quantity under consideration.

On the other hand, the Mandelstam formalism cannot be considered in general as a gauge-fixing and thus many standard techniques of QFT cannot be applied to this case. Independently of Tsamis and Woodard, Toller [1988] suggested to try to define the fields themselves with reference to geodesics and parallel transport, in order to give a direct physical significance to their correlation. This led to the classical radial gauge (Chapter 3), which, on account of its simplicity and of the field-potential inversion formula (Section 3.2), has proved to be useful for other purposes too (Sections 3.4, 3.5).

In this gauge the fields are defined in such a way that the vierbein $e_\mu^a(\xi)$ in a point ξ coincides with that which is parallel transported from the origin to ξ along a straight line. Moreover, this line is the (locally unique) geodesic which connects the origin to the point ξ and the geodesic distance from the origin is given by $|\xi|$. Thus in this gauge the correlation function $\langle e_\mu^a(0)e_\nu^b(\xi) \rangle$ automatically satisfies the criteria we enunciated at the end of Section 1.1.

Inserting the radial gauge fixing in some version of the functional integral of gravity, one may compute the corresponding propagator. This propagator gives automatically the lowest-order physical correlation function and allows in principle to compute this correlation to any order.

As expected, the propagator in the radial gauge is not trivial (including in fact the geodesics and parallel transport) and contains some singularities (Sections 4.2, 4.3). By means of a suitable projection procedure, however, it is possible to gain an insight into the singularities and to eliminate them. It is believed now that the “additional” divergences which appear in the mentioned work of Tsamis and Woodard could be eliminated through a similar procedure. There is no special connection between these divergences and the usual ultraviolet divergences of quantum gravity.

Summarizing, we have in the radial gauge a complete solution to the problem of computing perturbatively the vacuum correlations at geodesic distance. The limitations are those inherent to perturbation theory in quantum gravity (i.e., non renormalizability for the Einstein theory, or non unitarity for the $(R + R^2)$ -theory), and those due to the mathematical complexity of the radial propagator. On the other hand, some problems typical of algebraic gauges, like the need for special prescriptions in order to define uniquely the propagator [see for instance Bassetto, Nardelli and Soldati, 1991; Gaigg and Kummer, 1990], do not affect the radial gauge. Also, in four dimensions the four-particles interaction vertex is likely to vanish in radial gauge since the four fields should be at the same time orthogonal to ξ and between themselves.

1.3 Invariant gravitational correlations.

The formal advances in the determination of the radial propagators do not say us much on some physical features which characterize the gravitational vacuum correlations. In particular, we may ask: which are the main differences with respect to the other gauge fields? Which are the modes of the field which are really correlated? Is the gravitational interaction truly due to the exchange of virtual particles, and thus connected to the vacuum fluctuations in the same way like the other fundamental interactions do?

In Chapters 5 and 6, the first attempts to answer these questions are presented. The crucial quantities under investigation are the Wilson loops of the connection, which are scalar quantities and give a sensible information about the geometry of spacetime. The computations we present here start from a flat background and take into account corrections up to order \hbar . This is justified on length scales much larger than Planck's length; at smaller distances, non-perturbative methods must be employed (see Chapter 7).

In Chapter 5 the Wilson loops of the gravitational connection are defined (Section 5.1) and computed to order \hbar in Einstein's theory (Section 5.4). The invariant two-point functions of the Riemann curvature are computed too (Section 5.3).

The Wilson loops vanish and this peculiar result, when analyzed in terms of the relevant gauge group for the euclidean theory, $SO(4)$ (Section 6.1), shows that the functional integral of quantum gravity does not contain to order \hbar any configuration with localized curvature. Such a behaviour contrasts strongly with that of usual gauge fields, and may be interpreted as the absence, to this order, of true virtual gravitons (those which appear in the diagrams for the cross sections or similar represent excitations of the metric which do not carry curvature!). It can be shown that this result holds more generally than just for Einstein gravity, but relies on the flat background.

Chapter 6 contains also some more material on the Wilson loops: the correlation function of two loops to leading order and its “geodesic corrections” (Sections 6.2, 6.3). In Section 6.4 the formula for the static potential energy in quantum gravity is presented.

Finally, Chapter 7 is dedicated to lattice gravity, with special regard to the “quantum Regge calculus” of Hamber. The outcome of the recent Montecarlo simulations is very promising: the transition between the two phases of the lattice system appears to be of second order, and this has been confirmed by the behavior of the (simple) correlation functions which have been computed up to now.

Notations.

We refer mainly to spacetime dimension 4; otherwise the dimension is denoted by N and assumed to be bigger than 2. The units are such that $c = \hbar = 1$; in some formulas where the classical contributions need to be distinguished from the quantum ones, \hbar is written explicitly. The symbol κ (or sometimes ℓ) denotes Planck's length $\sqrt{16\pi G}$. In some of the reviewed papers the metric has euclidean signature, in other it is minkowskian; we have used both notations, although for gravity the correspondence between the euclidean theory and the theory on Minkowski space is not completely established yet [see for instance Mazur and Mottola, 1990].

The derivatives of the form ∂_μ are taken with respect to the whole argument of the function. A suffix like $[ab]$ denotes antisymmetrization (without the factor $\frac{1}{2}$). The notation “ $o(\)$ ” means “of higher order than”; “ $O(\)$ ” means “of the same order as”.

For recent general reviews on quantum gravity and references see for instance Alvarez [1989] or the lectures of the Les Houches Summer School, 1992.

2. THE MANDELSTAM COVARIANTS.

In this first, “historical” chapter we shall first introduce the general classical Mandelstam covariants according to the original definition [Mandelstam, 1962], with a special emphasis on their geometrical meaning (Section 2.1); then we shall present in short the perturbative computation, due to Tsamis and Woodard, of a special kind of two-point functions inspired to Mandelstam’s covariants (Section 2.2).

2.1 The original definition.

In the early Sixties it was already clear to a few quantum theorists that in a true theory of quantum gravity the gravitational field had to play the hard role of furnishing coordinates to spacetime while being at the same time a quantum object.

In the mentioned paper, Mandelstam discussed a technique for defining spacetime and its point without using coordinates, but replacing them with “paths”, which may be themselves influenced by the (eventually quantized) gravitational field.

His pioneering work, although technically heavy in some points, had a great influence on the subsequent developments of gauge theories. Also, it is very remarkable that through his paths-based formalism Mandelstam succeeded in [1968] in finding the correct Feynman rules for gravity and Yang-Mills theories; these rules were derived at the same time by De Witt [1967 b, c] and by Faddeev and Popov, whose functional-integral technique has become later the standard one.

In the Mandelstam formalism the coordinates of spacetime do not appear. A point is defined as the end of a path, which in turn is specified by an infinite list of “small steps” to be taken in sequence by an observer, with reference to his local frame. After each infinitesimal step, the frame is parallel transported to the new location, and so on.

Thus the fields are not functions of four coordinates (x_0, x_1, x_2, x_3) , but functionals of “paths” P . On the other hand, the paths P are purely geometric, intrinsic objects, while the coordinates (x_0, x_1, x_2, x_3) are subject in General Relativity to a vast class of transformations without change of the physical results.

The basic gravitational field is the anholonomic Riemann curvature $R_{abcd}(P)$. The components of R are referred to the parallel transported local reference frame, and are then coordinate-independent. The fields can be canonically quantized and their correlation functions defined, through a pretty complicated procedure.

In a recent work of Teitelboim [1993] a more accurate mathematical definition of the classical Mandelstam’s theory is given, also showing how the field equations in path space may be derived from an action principle and how is it possible to decide consistently “when two paths end at the same point” – a concept which was undefined in the original Mandelstam’s theory.

An interesting possibility within the Mandelstam formalism is that of fixing the paths through some prescription. A natural prescription, corresponding to straight lines in flat spacetime, is that of taking geodesic paths. In fact, the direction of a geodesic line, as observed in the free-falling local system, is constant, so the “infinite” list defining the path P is very much simplified. This geometrical construction is explicitly implemented – starting from a given central point – in the radial gauge (Chapters 3, 4), which thus

establishes a connection between a special case of Mandelstam formalism and the usual formalism of General Relativity or Einstein-Cartan theory.

2.2 The computation of Tsamis and Woodard.

The starting point of this work [1992] was that of considering the correlation of two Mandelstam-like fields, placed at the ends of a geodesic line of fixed length. This correlation was written by inserting the suitable matrices of parallel transport and evaluated perturbatively to one-loop order.

The motivation for this computation originated in the troubles of quantum gravity with the unrenormalizable ultraviolet divergences. Some parallels with QED suggested that these divergences could cancel when the true “physical Green’s functions” are computed, instead of generic quantities which are subject to coordinates transformations and are thus unphysical.

The paper of Tsamis and Woodard is very remarkable from the “phylosophical” point of view and for its quantum-field content.

“Instead of changing the fundamental dynamical principle”, they proposed “to alter the way in which physical questions are asked. This idea is hardly new, nor is it unique to quantum gravity [...]. The particular detail we propose to treat more carefully is the physical coordinate system. We shall do this by introducing a new interpolating field, the Mandelstam covariant [...]. The Mandelstam covariant is not a local, invertible redefinition of the usual field [...]. It follows that the Green’s functions deriving from the Mandelstam covariant need not interpolate the usual S-matrix, nor contain the usual ultraviolet divergences. Indeed we will prove that the usual result *does* change. Unfortunately it does so in the direction of greater divergence; however, the reason for this seems to be correctable”

And going on with the citation: “The issue for us is not so much that local Green’s functions depend upon the gauge but rather that this gauge dependence makes them unwieldy experiments. In fact none of them is a reliable probe of quantum gravity, because they call for measurements that can never actually be made [...] and must we be very concerned over the fact that the theory predicts a divergent result for them?”

Tsamis and Woodard write the Mandelstam covariant as

$$\mathcal{R}_{abcd}[e](V, x) = \bar{M}_a^k[e](V, x) \bar{M}_b^l[e](V, x) \bar{M}_c^m[e](V, x) \bar{M}_d^n[e](V, x) \bar{R}_{klmn}(\chi[e](1, V, x))$$

where the bar over the Riemann tensor denotes that its indices are local Lorentz:

$$\bar{R}_{abcd}(z) = e_{\alpha a}(z) e_b^{\beta}(z) e_c^{\gamma}(z) e_d^{\delta}(z) R_{\beta\gamma\delta}^{\alpha}(z).$$

By $\chi[e](\tau, V, x)$ the geodesic path is meant which carries the Mandelstam observer from his origin x to the point at which he measures the curvature. τ is the geodesic parameter and V the initial direction. It is always assumed that the fundamental dynamical variable is the vierbein e . M is the matrix of the parallel transport:

$$\begin{aligned} \bar{M}_b^a[e](V, x) &= e_{\alpha}^a(\chi(1)) M_{\beta}^{\alpha}[e](V, x) e_b^{\beta}(x) \\ M_{\beta}^{\alpha}[e](V, x) &= P \exp \left(- \int_0^1 d\tau \dot{\chi}^{\mu}(\tau) \Gamma_{\mu\beta}^{\alpha}(\chi(\tau)) \right). \end{aligned}$$

It can be proved that \mathcal{R} can not be related to R by any local field transformation. The reason for this is essentially the appearance in \mathcal{R} of the non-local matrix M .

The one-loop computation of the correlations of \mathcal{R} is extremely long. The expectation values are organized into 30 “ k -point functions” which were evaluated using the dimensional regularization. The “ V -ordering” corrections (corresponding to the path ordering) were then added, and the final result expressed as a linear combination of 8-index “master tensor structures”.

At each order the result breaks up into “uncorrected terms” (corresponding to the naive flat space perturbations), “length corrections”, “index corrections” and “length and index corrections”. This is precisely what is expected from the physical features of the gravitational correlation functions (see the Introduction, and Section 6.3 for a concrete example).

The “additional singularities” which affect the result have the same form like those of the “ $\langle P^0 P^0 \rangle$ propagator” of radial gauge. For this reason, it is very likely that they could be eliminated, in principle, by a suitable projection procedure (compare Section 4.2, 4.3). In practice, however, the algebraic structure of the Mandelstam covariant employed by Tsamis and Woodard is too complicated to allow this. For a comparison with the radial-gauge formalism, see the beginning of Section 4.4.

3. CLASSICAL RADIAL GAUGE AND APPLICATIONS.

In this chapter an introduction to the classical radial gauge is given. In Section 3.1 the gauge condition is defined (in the vierbein formalism) and its attainability is shown. In Section 3.2 we write the formulas which allow to express the radial vierbein and the radial connection in terms of the Riemann tensor and of the torsion (the so-called inversion formulas). In Section 3.3 the transformation properties of the radial fields under rotation or translation of the central vierbein are given. In Section 3.4 we illustrate in short a remarkable application of the radial gauge to (2+1)-gravity and finally, in Section 3.5, an application to the description of the geodesic motion of test particles in a fluctuating gravitational field.

3.1 Main features.

The radial gauge $x_\mu A_\mu(x) = 0$ has been proposed many years ago by Fock [1937] and Schwinger [1952, 1989; for a complete bibliography see M. and Toller, 1990] for electrodynamics or Yang-Mills theory. Its most interesting property is the so called fields-potentials inversion formula. Let us suppose that the field strength $F_{\mu\nu}(x)$ is known. In order to compute from $F_{\mu\nu}(x)$ the vector potential $A_\mu(x)$ in radial gauge, one must solve the system

$$\begin{aligned} x_\mu A_\mu(x) &= 0 \\ \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)] &= F_{\mu\nu}(x). \end{aligned}$$

Its general solution is

$$A_\mu(x) = \int_0^1 d\lambda \lambda x_\nu F_{\nu\mu}(\lambda x) + A_\mu^{Hom}(x),$$

where the homogeneous part $A_\mu^{Hom}(x)$ is solution of the system

$$\begin{aligned} x_\mu A_\mu^{Hom}(x) &= 0 \\ \partial_\mu A_\nu^{Hom}(x) - \partial_\nu A_\mu^{Hom}(x) + ig[A_\mu^{Hom}(x), A_\nu^{Hom}(x)] &= 0; \end{aligned}$$

that is, we have

$$A_\mu^{Hom}(x) = \frac{\partial f(x)}{\partial x^\mu}, \quad (3.1)$$

being f an arbitrary homogeneous function of degree 0. If one wants $A_\mu(x)$ to be regular at the origin, f must be set equal to zero. In this case, the gauge condition can be regarded as “complete”, in the sense that no residual gauge is present. Until the introduction of the general radial projectors (Section 4.2), we shall stay here in the hypothesis that all fields are regular at the origin; thus our inversion formula will be simply given by the “0-1 projector”

$$A_\mu(x) = \int_0^1 d\lambda \lambda x_\nu F_{\nu\mu}(\lambda x). \quad (3.2)$$

Let us now consider the case of gravitation. We recall that in order to describe the gravitational field it is sometimes convenient to use, instead of the metric tensor $g_{\mu\nu}(x)$, the vierbein fields $e_\mu^a(x)$ and $e_a^\mu(x)$, which are defined by

$$e_\mu^a(x) e_\nu^b(x) \delta_{ab} = g_{\mu\nu}(x); \quad (3.3)$$

$$e_a^\mu(x) e_\nu^a(x) = \delta_\nu^\mu; \quad e_a^\mu(x) e_\mu^b(x) = \delta_a^b. \quad (3.4)$$

The holonomic indices μ, ν, \dots and the “internal” indices a, b, \dots range, in general, between 1 and N , where N is the dimension of spacetime; the N vectors $\{e_1(x), \dots, e_N(x)\}$ can be thought to represent a local reference frame at any point x . Condition (3.3) determines $e_\mu^a(x)$ up to a local rotation; the gauge potentials Γ_μ^{ab} and field strengths $R_{\mu\nu}^{ab}$ corresponding to this local invariance are related to the Christoffel symbol and to the Riemann tensor by the formulas†

$$\Gamma_{\rho\sigma}^\mu = \Gamma_{b\sigma}^a e_\rho^b e_a^\mu + e_a^\mu \partial_\sigma e_\rho^a; \quad (3.5)$$

$$R_{\nu\rho\sigma}^\mu = e_a^\mu e_\nu^b R_{b\rho\sigma}^a. \quad (3.6)$$

In this formalism, two gauge fixing conditions are needed, in order to eliminate the freedom to make diffeomorphisms (i.e., coordinate transformations), and the freedom for local rotations of the vierbein. Recently a radial gauge condition has been proposed also for gravity. It has the form [M. and Toller, 1990]

$$\xi^\mu \Gamma_{b\mu}^a(\xi) = 0, \quad (3.7a)$$

$$\xi^\mu e_\mu^a(\xi) = \xi^\mu \delta_\mu^a. \quad (3.7b)$$

The gauge-fixing conditions (3.7a) and (3.7b) have a simple geometrical interpretation. Condition (3.7a) means that the tetrads are parallel transported from the origin along the straight lines of the form

$$\{s\xi, 0 \leq s \leq 1\}; \quad (3.8)$$

condition (3.7b) means that these lines are auto-parallel, i.e. they are geodesic lines (in the absence of torsion). The coordinates ξ coincide then with the well-known “normal coordinates” [see for ex. Kobayashi and Nomizu, 1969].

It is possible to give explicit formulae for the calculation of the normal coordinates and of the parallel-transported tetrads of the radial gauge. To this end, we consider an

† The connection $\Gamma_{b\mu}^a$ is viewed as a variable independent of the vierbein; so this formalism is often referred to as “first order”, while the metric formalism is defined to be of the “second order”. As we shall illustrate below, it is possible to define a radial gauge condition also in the second order formalism. We do not dwell here upon the relationship between the two formalisms and the role of the torsion [see, for instance, Hehl et al., 1976]. We notice, however, that the radial gauge in the first order formalism is meaningful also in spaces without metric connection and for gauge groups different from the Poincaré group.

arbitrary coordinate system x^μ and we indicate by x_0^μ the coordinates of the origin P_0 . We assume that the holonomic coefficients of the connection $\hat{\Gamma}_{\nu\rho}^\mu(x)$ are known. We indicate by \hat{e}_a^μ the components of the tetrads with respect to the natural holonomic basis determined by the coordinates x^μ . The components of the dual tetrads in the natural basis determined by the normal coordinates ξ^μ can be computed by means of the formula

$$e_\mu^a(\xi) = \frac{\partial x^\rho(\xi)}{\partial \xi^\mu} \hat{e}_\rho^a(\xi).$$

Since the tetrads are parallel-transported along the line (3.8), we have

$$\frac{d\hat{e}_b^\mu(\lambda)}{d\lambda} = -\hat{\Gamma}_{\sigma\tau}^\mu[x(\lambda)] \hat{e}_b^\sigma(\lambda) \hat{e}_a^\tau(\lambda) v^a \quad (3.9)$$

and the fact the the vector with constant anholonomic components v^a is tangent to the line (3.8) is expressed by the equation

$$\frac{dx^\mu(\lambda)}{d\lambda} = \hat{e}_a^\mu(\lambda) v^a. \quad (3.10)$$

The equations (3.9) and (3.10), with the initial conditions $x^\mu(0) = x_0^\mu$ and $\hat{e}_a^\mu(0) = e_a^\mu(P_0)$, determine the quantities x^μ and \hat{e}_b^μ as functions of λ and of v^a . However, it is easy to see that they depend on a particular combination of these variables, namely on the normal coordinates $\xi^a = \lambda v^a$. If we are considering a metric space and a metric connection, we can choose an orthonormal initial tetrad $e_a(0)$ and all the parallel-transported tetrads are automatically orthonormal.

It is convenient to transform these differential equations with their initial conditions into the following pair of coupled integral equations

$$x^\mu(\xi) = x_0^\mu + \xi^a \int_0^1 \hat{e}_a^\mu(\lambda \xi) d\lambda, \quad (3.11)$$

$$\hat{e}_b^\mu(\xi) = \hat{e}_b^\mu(0) - \xi^a \int_0^1 \hat{\Gamma}_{\sigma\tau}^\mu[x(\lambda \xi)] \hat{e}_b^\sigma(\lambda \xi) \hat{e}_a^\tau(\lambda \xi) d\lambda. \quad (3.12)$$

It is possible to solve these equations perturbatively to any desired order in $\hat{\Gamma}$, by substituting at each step the lower order solution in the left hand side integrals. This shows that the gauge condition is “attainable” in the sense that given the fields in an arbitrary gauge, it is always possible to compute the corresponding fields in the radial gauge. A simple application of these equations is the computation to lowest order in $\hat{\Gamma}$ of the coordinate transformation $x^\mu(\xi)$. One obtains

$$x^\mu(\xi) = \xi^\mu - \xi^\rho \xi^\sigma \int_0^1 dt (1-t) \hat{\Gamma}_{\rho\sigma}^\mu(t\xi) + o(\hat{\Gamma}^2). \quad (3.13)$$

This formula will be used in Section 3.5 and 6.3.

Note that if we change the initial conditions by performing a Lorentz transformation L of the tetrad $e_a(0)$, (or more generally a transformation of the gauge group) we get a new solution obtained from the old one by means of the same Lorentz transformation L applied to all the tetrads and to the normal coordinates. This property is not trivial, because the parallel transport is generally non-commutative. A translation of the origin P_0 affects the normal coordinates and the parallel-transported tetrads in a more complicated way, which will be illustrated in Section 3.3.

The radial gauge conditions (3.7a) and (3.7b) can be regarded, in a sense, as an operational prescription which permits one to locate the measuring instruments in a neighbourhood of the observer, who lies at the origin P_0 . In fact a simple way to explore this neighbourhood is to send from the origin many “space-probes” carrying clocks, gyroscopes and measuring instruments. A space-probe will be launched with four-velocity v^a with respect to the given tetrad $e_a(0)$ and, if τ is the proper time measured by the clock, $\xi^a = \tau v^a$ are the normal coordinates (in the absence of torsion). Of course, in Minkowski spacetime only the interior of the future cone can be explored in this way.

Any space-probe will be able, in principle, to measure by local experiments the anholonomic components of the fields at any point ξ . Moreover, let us go on with our *Gedankenexperiment* and suppose that any space-probe emits all the time some peculiar light signals along the axis of its local frame; the signals are received and recorded by the neighboring space-probes. By collecting all the data, the mentioned “central observer” will thus be able to compute also the vierbein $e_\mu^a(\xi)$.

Finally, we remind that the radial gauge can be introduced also in the so-called second order formalism, that is, as a condition on the metric tensor [Menotti, M. and Seminara, 1993]. In this case it is defined by

$$\xi^\mu g_{\mu\nu}(\xi) = \xi^\mu \eta_{\mu\nu}. \quad (3.14)$$

A number of properties of the radial gauge in the tetrad formalism still hold in the metric formalism. We enumerate them without proof:

- if $g_{\mu\nu}(\xi)$ is regular at the origin, then we have $g_{\mu\nu}(0) = \eta_{\mu\nu}$;
- coordinates ξ satisfying (3.14) are normal coordinates;
- the gauge is attainable, under some regularity hypothesis.

An equation analogous to the inversion formulas (3.17), (3.19) holds just in the linearized theory. If we denote by R^L the linearized Riemann tensor, we have

$$\begin{aligned} h_{\nu\beta}(x) &= -2 x^\alpha x^\mu \int_0^1 d\tau \tau \int_0^1 d\lambda \lambda^2 R_{\mu\nu\alpha\beta}^L(\lambda\tau x) = \\ &= -2 x^\alpha x^\mu \int_0^1 d\lambda \lambda(1-\lambda) R_{\mu\nu\alpha\beta}^L(\lambda x), \end{aligned}$$

provided $|R^L(x)| < |x|^{-2+\varepsilon}$ as $x \rightarrow 0$.

3.2 Inversion formulas.

We shall now derive from the radial gauge conditions (3.7a) and (3.7b) two formulas analogous to (3.2), which give the field potentials $\Gamma_{b\mu}^a(\xi)$ and $e_\mu^a(\xi)$ in terms of the Riemann tensor $R_{b\mu\nu}^a(\xi)$ and the torsion tensor $S_{\mu\nu}^a(\xi)$.

We would like first to make the following observation. Remember that in this chapter we work in the hypothesis that the fields are regular (and differentiable) at the origin. Thus, taking the derivative of eq.s (3.7a) and (3.7b) we obtain

$$\Gamma_{b\mu}^a(0) = 0, \quad e_\mu^a(0) = \delta_\mu^a.$$

These equations show that in the radial gauge the gauge potentials at the origin take the values they have in a flat space. In other words, it is possible to eliminate the gravitational field at a given point. This is a formulation of the equivalence principle which is valid also in the presence of torsion, when it is not possible to eliminate the holonomic connection coefficients at a given point [see the references in M. and Toller, 1990].

Actually, one of the motivations which led to the formulation of the radial gauge was the need of a generalization of the equivalence principle to 10-dimensional theories defined on group manifolds. To this end, it is crucial for the gauge to be “local”, in the sense that the inversion formulas should involve only the fields in a neighbourhood of the origin. The formulas of this subsection match this condition. In the following chapters we shall see that, giving up the regularity of the fields at the origin and exploiting the residual gauge, it is possible to write down some formulas which are similar to those of this chapter, but contain integrals on the domain $(1-\infty)$. In that case, the “locality” of the gauge condition is lost.

We remind that the Riemann tensor and the torsion tensor are given by the expressions

$$R_{b\mu\nu}^a = \partial_\mu \Gamma_{b\nu}^a - \partial_\nu \Gamma_{b\mu}^a + \Gamma_{c\mu}^a \Gamma_{b\nu}^c - \Gamma_{c\nu}^a \Gamma_{b\mu}^c, \quad (3.15)$$

$$S_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + e_\nu^b \Gamma_{b\mu}^a - e_\mu^b \Gamma_{b\nu}^a. \quad (3.16)$$

Multiplying (3.15) by ξ^μ we have from (3.7a)

$$\xi^\mu R_{b\mu\nu}^a(\xi) = \xi^\mu \partial_\mu \Gamma_{b\nu}^a(\xi) + \Gamma_{b\nu}^a(\xi).$$

If we now put $\xi \rightarrow \lambda \xi$, we obtain

$$\frac{d}{d\lambda} (\lambda \Gamma_{b\nu}^a(\lambda \xi)) = \lambda \xi^\mu R_{b\mu\nu}^a(\lambda \xi)$$

and integrating we finally have

$$\Gamma_{b\nu}^a(\xi) = \xi^\mu \int_0^1 R_{b\mu\nu}^a(\lambda \xi) \lambda d\lambda. \quad (3.17)$$

In a similar way, by multiplying (3.15) by ξ^μ and taking into account condition (3.7b) we obtain

$$\xi^\mu S_{\mu\nu}^a(\xi) = \xi^\mu \partial_\mu e_\nu^a(\xi) + e_\nu^a(\xi) - \delta_\nu^a - \xi^b \Gamma_{b\nu}^a(\xi);$$

by the same procedure we obtain

$$\frac{d}{d\lambda} (\lambda(e_\nu^a(\lambda\xi) - \delta_\nu^a)) = \lambda\xi^b \Gamma_{b\nu}^a(\lambda\xi) + \lambda\xi^\mu S_{\mu\nu}^a(\lambda\xi)$$

and

$$e_\nu^a(\xi) = \delta_\nu^a + \int_0^1 [\xi^b \Gamma_{b\nu}^a(\lambda\xi) + \xi^\mu S_{\mu\nu}^a(\lambda\xi)] \lambda d\lambda. \quad (3.18)$$

By substituting (3.17) into (3.18) we have

$$e_\nu^a(\xi) = \delta_\nu^a + \xi^\mu \xi^b \int_0^1 R_{b\mu\nu}^a(\lambda\xi) (1 - \lambda) \lambda d\lambda + \xi^\mu \int_0^1 S_{\mu\nu}^a(\lambda\xi) \lambda d\lambda. \quad (3.19)$$

The equations (3.17) and (3.19) are the analogue of (3.2). The gauge potentials they give satisfy the gauge conditions (3.7a) and (3.7b) as a consequence of the antisymmetry of $R_{b\mu\nu}^a$ and $S_{\mu\nu}^a$ with respect to the indices μ, ν . However, they solve (3.15) and (3.16) only if the functions $R_{b\mu\nu}^a(\xi)$ and $S_{\mu\nu}^a(\xi)$ satisfy some conditions. These conditions have been derived and used in the Yang-Mills case [Durand e Mendel, 1982] and a similar treatment can be given in the case under consideration. If we substitute (3.17) and (3.19) into (3.15), (3.16), after a long calculation we find that they are equivalent to the following projected Bianchi identities

$$\begin{aligned} \xi^\mu \sum_{\{\mu\nu\rho\}} (\partial_\mu R_{b\nu\rho}^a + \Gamma_{c\mu}^a R_{b\nu\rho}^c - \Gamma_{b\mu}^c R_{c\nu\rho}^a) &= 0, \\ \xi^\mu \sum_{\{\mu\nu\rho\}} (\partial_\mu S_{\nu\rho}^a + \Gamma_{b\mu}^a S_{\nu\rho}^b - e_\mu^b R_{b\nu\rho}^a) &= 0, \end{aligned}$$

in which the potentials e_μ^b and $\Gamma_{b\mu}^a$ are replaced by the integral expressions (3.17), (3.19). We have indicated by $\sum_{\{\mu\nu\rho\}}$ the sum over the cyclic permutations of the indices μ, ν, ρ .

These conditions have a non local character, since they are expressed by integro-differential equations. Also the Einstein field equations $e_a^\mu R_{b\mu\nu}^a = 0$ contain the potential e_μ^a and therefore take a non local character if we want to express them in terms of the curvature alone. In conclusion, if we try to use the inversion formulas to eliminate the gauge potentials from the theory (like in Mandelstam's formalism), we get very complicated non local field equations.

3.3 Translation of the origin.

If we adopt the radial gauge, the geometry of the space-time manifold in a neighbourhood of the origin P_0 is completely described by the functions $e_\nu^a(\xi)$, $\Gamma_{b\nu}^a(\xi)$, which have to satisfy the gauge conditions (3.7a), (3.7b) and the field equations of the theory we are considering. In a similar way, the matter fields are completely described by their components with respect to the tetrads, expressed as functions of the normal coordinates

ξ . The only arbitrary elements in this description are the choice of the origin P_0 and of the tetrad $e_a(P_0)$. Thus the fields in the radial gauge can be considered as “observable” as the fields in Minkowski space-time described by their components with respect to a given inertial coordinate frame. In fact, also in this case the values of the field components depend on the choice of the origin and of the directions of the coordinate axes.

The transformation properties of the fields with respect to the Poincaré group permit to compute the components in the new reference frame in terms of the old components. For instance, if V^a is a vector field, we have

$$V'^a(x') = [L^{-1}]_b^a V^b(x), \quad x = Lx' + x_0. \quad (3.20)$$

In the following we generalize this formula to a curved space-time with radial gauge, namely we derive the explicit form of the transformations of the fields caused by a translation of the origin. We consider a radial gauge with origin at the point P_0 , coordinates ξ^μ and tetrads $e_a(\xi)$ and we start from the point P_1 with coordinates ξ_1^μ and from the tetrad

$$e'_a(0) = L_a^b e_b(\xi_1)$$

to build a new radial gauge with coordinates ξ'^μ and tetrads $e'_a(\xi')$. If, for simplicity, we take $L = 1$, the coordinates and the tetrads are connected by

$$\xi^\mu = \Xi^\mu(\xi_1, \xi'), \quad e'_a(\xi') = \Omega_a^b(\xi_1, \xi') e_b(\xi).$$

It is easy to see that for general values of L these relations are modified as follows

$$\xi^\mu = \Xi^\mu(\xi_1, L\xi'), \quad e'_a(\xi') = L_a^b \Omega_b^c(\xi_1, L\xi') e_c(\xi).$$

The transformation property of a vector field is

$$V'^a(\xi') = [\Omega^{-1}(\xi_1, L\xi') L^{-1}]_b^a V^b(\xi) \quad (3.21)$$

and tensors of arbitrary order transform in a similar way. In Poincaré or Euclidean gauge theories it is easy to write also the transformation properties of spinor fields. In a flat space-time we have

$$\Xi^\mu(\xi_1, \xi') = \xi_1^\mu + \xi'^\mu, \quad \Omega_a^b(\xi_1, \xi') = \delta_a^b \quad (3.22)$$

and the transformation property (3.21) takes the form (3.20). It follows from the definitions that (3.22) holds for general spaces when ξ_1 and ξ' are proportional.

The gauge potentials, which describe the geometry, transform in the following way (for $L = 1$):

$$e'_\mu(\xi') = \frac{\partial \Xi^\nu(\xi_1, \xi')}{\partial \xi'^\mu} [\Omega^{-1}(\xi_1, \xi')]_c^a e_c^b(\xi), \quad (3.23)$$

$$\Gamma'^a_{b\mu}(\xi') = [\Omega^{-1}(\xi_1, \xi')]_c^a \left[\Omega_b^d(\xi_1, \xi') \frac{\partial \Xi^\nu(\xi_1, \xi')}{\partial \xi'^\mu} \Gamma_{d\nu}^c(\xi) + \frac{\partial \Omega_b^c(\xi_1, \xi')}{\partial \xi'^\mu} \right]. \quad (3.24)$$

Note that these transformations do not form a group because the quantities $\Xi^\mu(\xi_1, \xi')$ and $\Omega_a^b(\xi_1, \xi')$ can depend also on the initial point P_0 . They can be computed by means of a method similar to that used in Section 3.1. The result are the following integral equations

$$\begin{aligned}\Xi^\nu(\xi_1, \xi') &= \xi_1^\nu + \xi'^a \int_0^1 \Omega_a^b(\xi_1, \lambda \xi') e_b^\nu(\Xi(\xi_1, \lambda \xi')) d\lambda, \\ \Omega_a^b(\xi_1, \xi') &= \delta_a^b + \\ &\quad - \xi'^d \int_0^1 \Omega_d^e(\xi_1, \lambda \xi') e_e^\nu(\Xi(\xi_1, \lambda \xi')) \Gamma_{cv}^b(\Xi(\xi_1, \lambda \xi')) \Omega_a^c(\xi_1, \lambda \xi') d\lambda.\end{aligned}$$

It is useful to consider an infinitesimal displacement ξ_1 of the origin and put

$$\begin{aligned}\Xi^\nu(\xi_1, \xi') &= \xi_1^\nu + \xi'^\nu + \xi_1^\mu A_\nu^\mu(\xi') + O(\xi_1^2), \\ \Omega_a^b(\xi_1, \xi') &= \delta_a^b + \xi_1^\mu B_{a\mu}^b(\xi') + O(\xi_1^2).\end{aligned}$$

From eq. (3.22), which holds when ξ_1 and ξ' are proportional, we get the conditions

$$\xi^\mu A_\mu^\nu(\xi) = 0, \quad \xi^\mu B_{a\mu}^b(\xi) = 0.$$

Formulas (3.21), (3.23) and (3.24) take the form

$$V'^a(\xi) - V^a(\xi) = \xi_1^\rho \delta_\rho V^a + O(\xi_1^2), \quad e'_\mu^a(\xi) - e_\mu^a(\xi) = \xi_1^\rho \delta_\rho e_\mu^a + O(\xi_1^2),$$

$$\Gamma'_{b\mu}^a(\xi) - \Gamma_{b\mu}^a(\xi) = \xi_1^\rho \delta_\rho \Gamma_{b\mu}^a + O(\xi_1^2),$$

where we have introduced suitable definitions for $\delta_\rho V^a$, $\delta_\rho e_\mu^a$, $\delta_\rho \Gamma_{b\mu}^a$ [M. and Toller, 1990]; for instance,

$$\delta_\rho V^a = -B_{b\rho}^a V^b + (\delta_\rho^\nu + A_\rho^\nu) \frac{\partial V^a}{\partial \xi^\nu}.$$

We may easily obtain integral equations for the quantities A and B , which can be solved perturbatively for small values of $e_a^\mu - \delta_a^\mu$ and $\Gamma_{a\mu}^b$. The formulas obtained in this subsection turn also out to be useful in applying the radial gauge to the problem of non-singular homogeneous and isotropic random fields.

3.4 Solutions of (2+1)-gravity.

In this section we give a brief account of the applications of the radial gauge to (2+1)-gravity due to Menotti and Seminara.

In the last years the amount of field-theoretical work about 2- and 3-dimensional models has greatly increased. The motivations for this interest are various. In general, lower dimensional theories are more easily solved in a complete and rigorous mathematical fashion than higher dimensional ones. Some important and general effects, like spontaneous

breaking of gauge symmetry, anomalies, solitons, have first been discovered in two dimensions. For another class of phenomena, the generalization to higher dimensions is very difficult or impossible; there exists, however, some real physical system to which the theory can be applied. This is the case, for instance, of the high-temperature superconductivity and the quantum Hall effect.

In the case of (2+1)-dimensional gravity† the main interest of the theory resides in its connection to the topological theories and to the problem of the cosmic strings in 4 dimensions [Vilenkin, 1985]. Starting from the work of Deser, Jackiw and 't Hooft [1984] a number of papers has been devoted to the classical solutions in (2+1) dimensions, and some new solution techniques have been developed [Clement, 1985, 1990; Deser and Jackiw, 1989; Grignani and Lee, 1989].

All the known solutions, and a lot of new ones, have been re-obtained by Menotti and Seminara [1991 a, 1992] using the radial gauge. In fact, we recall that in radial gauge the tetrads and the connection can be written – in the absence of torsion – as integrals of the Riemann's tensor $R_{\mu\nu}^{ab}$ (eq.s (3.17), (3.19)). Moreover, in (2+1) dimensions the Einstein's equations state a linear dependence between the tensor $R_{\mu\nu}^{ab}$ and the energy-momentum tensor T_a^μ ‡. If the energy-momentum tensor is given, we can therefore write the metric as an integral of T_a^μ . Of course, in order the whole procedure to be consistent, T_a^μ cannot be arbitrary; it must satisfy a constraint relation which is related to the Bianchi identities. We shall derive it here following a slightly different approach from that of Menotti and Seminara. Let us consider the Einstein action in the first-order formalism, in any dimension $N \geq 3$:

$$S = \text{const.} \int d^N x R_{\mu\nu}^{ab} e_{\rho_1}^{c_1} e_{\rho_2}^{c_2} \dots e_{\rho_{N-3}}^{c_{N-3}} e_\sigma^d \varepsilon_{abc_1c_2\dots c_{N-3}d} \varepsilon^{\mu\nu\rho_1\rho_2\dots\rho_{N-3}\sigma},$$

where $R_{\mu\nu}^{ab}$ is defined in terms of the connection Γ_μ^{ab} like in (3.15). Coupling the vierbein to an energy-momentum source T_d^σ and minimizing S we obtain Einstein's equations in the form

$$R_{\mu\nu}^{ab} e_{\rho_1}^{c_1} \dots \varepsilon_{abc_1c_2\dots c_{N-3}d} \varepsilon^{\mu\nu\rho_1\rho_2\dots\rho_{N-3}\sigma} = \text{const. } T_d^\sigma,$$

or

$$R_{\mu\nu}^{ab} e_{\rho_1}^{c_1} \dots \varepsilon_{abc_1c_2\dots c_{N-3}d} \varepsilon^{\mu\nu\rho_1\rho_2\dots\rho_{N-3}\sigma} e_{\sigma e} = \text{const. } T_{de}. \quad (3.25a)$$

Now, it is generally believed † that in gravity it is not possible to “give the conserved source T_d^σ and solve for the fields e_μ^a and Γ_μ^{ab} ”. In fact, the conservation of T_d^σ cannot be checked without any knowledge of the geometry. So in general one must consider the simultaneous

† For an introduction to the classical and first-quantization aspects see Jackiw [1989]; for the second-quantization aspects see Witten [1988], and references.

‡ Because of this, there is no field propagation in empty space. This shows that (2+1)-gravity is physically very different from the “true” (3+1)-dimensional gravity.

† See, however, the paper of Boulware e Deser [1976].

evolution of the fields and the sources must (ADM formalism). Let us then look at the “inverse problem”: given the field, we want to find the conserved sources that produce it. By virtue of the Bianchi identities, the tensor T_d^σ defined by (3.25a) is covariantly conserved. We just need to impose the symmetry, obtaining the constraint

$$\{R_{\mu\nu}^{ab} e_{\rho_1}^{c_1} \dots e_{abc_1c_2\dots c_{N-3}d} \varepsilon^{\mu\nu\rho_1\rho_2\dots\rho_{N-3}\sigma} e_{\sigma e}\}_{[de]} = 0. \quad (3.25b)$$

The task of finding solutions of (3.25b) in any dimension is very difficult. Let us take, however, $N = 3$; eq. (3.25b) becomes

$$\{R_{\mu\nu}^{ab} \varepsilon_{abc} \varepsilon^{\mu\nu\rho} e_{\rho d}\}_{[cd]} = 0. \quad (3.26)$$

We now fix the radial gauge. The most general structure of a radial connection is

$$\Gamma_\mu^{ab}(\xi) = \varepsilon^{abc} \varepsilon_{\mu\rho\sigma} \xi^\rho A_c^\sigma(\xi), \quad (3.27)$$

where A_c^σ is an arbitrary tensor field. From (3.27) we have

$$R_{\mu\nu}^{ab} \varepsilon_{abc} \varepsilon^{\mu\nu\rho} = 2 A_c^\rho + \xi^\mu \partial_\mu A_c^\rho - \xi^\rho \left(\partial_\mu A_c^\mu - \frac{1}{2} \varepsilon_{abc} \varepsilon_{\alpha\mu\nu} \xi^\alpha A^{\mu a} A^{\nu b} \right).$$

We recall (eq. (3.18)) that in radial gauge the vierbein can be expressed as

$$e_{\rho d}(\xi) = \delta_{\rho d} + \int_0^1 d\lambda \lambda \xi_c \Gamma_{\rho d}^c(\lambda \xi).$$

Then the symmetry condition (3.26) becomes

$$\begin{aligned} & \varepsilon_{ab\rho} \left[2 A^{b\rho} + \xi^\mu \partial_\mu A^{b\rho} - \xi^\rho \left(\partial_\mu A^{b\mu} - \frac{1}{2} \varepsilon^{mn b} \varepsilon_{\sigma\mu\nu} \xi^\sigma A_m^\mu A_n^\nu \right) \right] + \\ & + \varepsilon_{\rho\sigma\nu} \xi^\sigma (2 A^{b\rho} + \xi^\mu \partial_\mu A^{b\rho}) \int_0^1 d\lambda \lambda^2 (\xi_a A_b^\nu(\lambda \xi) - \xi_b A_a^\nu(\lambda \xi)) = 0. \end{aligned} \quad (3.28)$$

This is the constraint equation given by Menotti and Seminara.

Some solutions of the constraint in (2+1) dimensions.

Let us first define the scalar operators

$$D_n = \xi_\mu \frac{\partial}{\partial \xi^\mu} + n; \quad D_\chi = \chi^\mu \frac{\partial}{\partial \xi^\mu}.$$

The operator D_n is invertible, for $n > 0$, provided it acts on functions which are regular at the origin. In fact, if $D_n f = 0$, then f is an homogeneous function of degree $-n$. The inverse of D_n has the form

$$D_n^{-1} f(\xi) = \int_0^1 d\lambda \lambda^{n-1} f(\lambda \xi). \quad (3.29)$$

We shall consider two simple “Ansaezte” for A_μ^a [Menotti and Seminara, 1991 a], which make vanish the term with the integral of the constraint (3.28) † :

- (1) $A_\mu^a(\xi) = \xi^a f_\mu(\xi)$, with $f_\mu(\xi)$ arbitrary;
- (2) $A_\mu^a(\xi) = \chi_\mu \rho^a(\xi)$, with $\rho^a(\xi)$ arbitrary.

Substituting the first ansatz into (3.28) we obtain

$$D_4 f^\rho(\xi) = \xi^\rho F(\xi), \quad (3.30)$$

where F is any function. From (3.29), (3.30) it follows that $f^\rho(\xi)$ is proportional to ξ^ρ ; we then conclude from (3.27) that the connection is identically zero. So the first Ansatz has to be rejected.

Substituting the second Ansatz, we get the equation

$$\chi^\mu D_2 \rho^a - \xi^\mu D_\chi \rho^a = \chi^a D_2 \rho^\mu - \xi^a D_\chi \rho^\mu. \quad (3.31)$$

Multiplying both sides of (3.31) by χ_μ we obtain (if $\chi^2 \neq 0$) the vector equation

$$\left[\chi^2 D_2 - (\vec{\chi} \vec{\xi}) D_\chi \right] \vec{\rho} = \vec{\chi} D_2 (\vec{\chi} \vec{\rho}) - \vec{\xi} D_\chi (\vec{\chi} \vec{\rho}).$$

In a suitable coordinate system, the operator in square brackets becomes an operator of the form D_2 , and is thus invertible; it is sufficient to choose as new coordinates $z_1 = (\vec{\chi} \vec{\xi})$, $z_2 = (\vec{a} \vec{\xi})$, $z_3 = (\vec{b} \vec{\xi})$, being a and b two vectors which are orthogonal to χ and between themselves. We then obtain for ρ the structure

$$\vec{\rho} = f \vec{\chi} + g \vec{\xi}. \quad (3.32)$$

It can be shown [Menotti and Seminara, 1991 a] that ρ has the same structure also in the case $\chi^2 = 0$. Substituting (3.32) in (3.31) we obtain the following differential equation for the functions f and g :

$$D_4 g = -D_\chi f \quad (3.33)$$

A simple and interesting solution of (3.33) can be found by setting

$$g = 0; \quad f = f \left[(\vec{a} \vec{\xi}), (\vec{b} \vec{\xi}) \right],$$

where a and b are two vectors orthogonal to χ . It can be easily shown that the corresponding vierbein is given by

$$e_\mu^a(\xi) = \delta_\mu^a + \frac{1}{2} \ell^2 \varepsilon^{abc} \varepsilon_{\alpha\mu\nu} \xi_b \xi^\nu \chi_c \chi^d \phi(\xi),$$

† In the absence of this term, the holonomic index of T is lowered, in practice, just with δ_μ^a instead of the whole vierbein; this could be viewed as a linear approximation. In fact, the other quadratic term in A vanishes too.

where

$$\phi(\xi) = \int_0^1 d\lambda \lambda (1 - \lambda) f(\lambda \xi).$$

From this, through (3.3), we obtain the metric

$$g_{\mu\nu}(\xi) = \eta_{\mu\nu} + \ell^2 \varepsilon_{\rho\sigma\mu} \varepsilon_{\alpha\beta\nu} \chi^\rho \xi^\sigma \chi^\alpha \xi^\beta \phi(\xi) \times \left\{ 1 + \frac{1}{4} \ell^2 [(\xi^\beta \xi_\beta) (\chi^\gamma \chi_\gamma) - (\xi^\gamma \chi_\gamma)^2] \phi(\xi) \right\}. \quad (3.34)$$

It is easy to verify that χ is a Killing vector for the metric (3.34). We thus go over to the reference system where χ assumes its simplest form. Let us consider the cases of time-like χ and space-like χ . (For the case of light-like χ see [Menotti and Seminara, 1991 a].)

(1) χ *time-like*. Setting $\chi = (1, 0, 0)$ the metric becomes

$$g_{00} = 1; \quad g_{0i} = 0;$$

$$g_{ij}(\xi) = -\delta_{ij} \pm \ell^2 \xi_i \xi_j \phi(\xi) \left(1 - \frac{1}{4} \ell^2 (\xi_1^2 + \xi_2^2) \phi(\xi) \right),$$

where the + sign holds for the diagonal and the - sign for the off-diagonal components. Going over to polar coordinates we obtain

$$g_{\mu\nu} = \text{diag} \left\{ 1, -1, -r^2 \left[1 - \frac{1}{2} \ell^2 r^2 \tilde{\phi}(r, \theta) \right]^2 \right\},$$

where

$$\tilde{\phi}(r, \theta) = \int_0^1 d\lambda \lambda (1 - \lambda) f(\lambda r \cos \theta, \lambda r \sin \theta). \quad (3.35)$$

Eq. (3.35) generalizes the usual conical metric to an extended distribution of matter which does not possess axial symmetry. Defining the defect angle by the ratio between the radius and the circumference, we obtain

$$\Delta\theta = \frac{1}{2} \ell^2 M,$$

where

$$M = \int_0^{2\pi} d\theta \int_0^\infty dr r f(r \cos \theta, r \sin \theta) = \int d^2x \sqrt{g} T_{00}.$$

Suitably choosing f we obtain the static point-like solutions of Deser, Jackiw and 't Hooft [1984] and the string solutions of Grignani and Lee [1989], Clement [1985, 1990], Deser and Jackiw [1989].

(2) χ *space-like*. Setting $\chi = (0, 0, 1)$ and performing the hyperbolic transformation

$$\xi^1 = S \cosh T; \quad \xi^0 = S \sinh T,$$

the metric takes the form

$$g_{\mu\nu} = \text{diag} \left\{ \mp S^2 \left[1 \mp \frac{1}{2} \ell^2 S^2 \tilde{\phi}(S, T) \right]^2, -1, -1 \right\},$$

where the $-$ sign holds inside the light cone, and the $+$ sign outside. This is a generalization of Rindler's metric.

The formulas we have written in this section are just an example of the solutions that can be found using the radial gauge. Moreover, it is possible to define a "reduced radial gauge" [Menotti and Seminara, 1991 a], which is useful in the analysis of stationary problems and allows a complete solution of the problem of timelike closed curves in (2+1)-gravity [Menotti and Seminara, 1993]. It is even possible to solve the constraint equation (3.28) in general form [Menotti and Seminara, 1992].

3.5 Motion of test particles in a fluctuating field.

In this section we shall present another application of the radial gauge, namely to the case of test particles moving in a weak quantized gravitational field (in four dimensions).

We start recalling that, from an operational point of view, the definition of Minkowski space-time is based on the possibility of building up an orthogonal network of rods and clocks. If a gravitational field is present this possibility is lost, but the equivalence between inertial and gravitational mass still allows a geometrical formulation of the theory. The idea that fields and geometry are intimately related has proved to be one of the most useful and fruitful concepts of physics. We shall thus assume that in the presence of a *classical* gravitational field space-time is described by a differentiable manifold M , endowed with a metric g .

We recall that it is possible to define on M a "geodesic structure" [see for ex. Kobayashi and Nomizu, 1969]. This structure is in some sense the physical manifestation of the geometrical framework of the theory: namely it is invariant with respect to coordinate transformations, and it can be operationally tested by observing the trajectories of many free-falling test particles. Owing to the equivalence principle, the mass of such particles has no importance, as long as they do not perturbate the field. The motion of test particles in an external field has been studied in its various aspects by many authors [see for instance Papapetrou, 1951; Souriau, 1974; Toller, 1983; Toller and Vaia, 1984].

In this section we want to show – as an application of the radial gauge – that a weak quantized gravitational field introduces a difficulty in the definition of the geodesic structure of space-time at very small distances: namely the vacuum correlations of the

field will influence the motion of a test particle, and this influence will depend on the *size* of the particle itself. Such a phenomenon is not difficult to be intuitively understood. In fact, the size L of a particle is most properly defined by diffraction experiments. This means that when the particle travels in a fluctuating gravitational field its motion will be affected just by the fluctuations of wavelenght λ such that $\lambda > L$. The point we would like to clarify is the average effect of these fluctuations on the geodesic structure measured by the particle. The conclusion is that such an effect is non vanishing and proportional to ℓ_{Planck}/L [M., 1992c]; this confirms that a geometric theory based on the equivalence principle is not operationally meaningful at distances comparable with ℓ_{Planck} , since the geometry depends on the size of the test particles.

The calculation relies on eq. (3.19), which gives the tetrad $e_\mu^a(\xi)$ in radial gauge as an integral of $R_{b\mu\nu}^a(\xi)$ (we assume that no torsion is present). Writing $R_{b\mu\nu}^a(\xi)$ in terms of the Riemann's tensor through (3.6), we obtain the equation

$$e_\nu^a(\xi) = \delta_\nu^a + \xi^\mu \xi^\beta \int_0^1 ds s(1-s) R_{\sigma\mu\nu}^\rho(s\xi) e_\rho^a(s\xi) e_b^\sigma(s\xi).$$

This is an integral equation for $e_\nu^a(\xi)$, which can be iteratively solved to the desired order in R . After two iterations we find

$$\begin{aligned} e_\mu^a(\xi) = & \delta_\mu^a + \delta_\nu^a \xi^\alpha \xi^\beta \int_0^1 ds s(1-s) R_{\alpha\beta\mu}^\nu(s\xi) + \\ & + \delta_\nu^a \xi^\alpha \xi^\beta \xi^\rho \xi^\sigma \int_0^1 ds s^3(1-s) \int_0^1 dt (1-t) R_{\alpha\beta\gamma}^\nu(s\xi) R_{\rho\sigma\mu}^\gamma(st\xi) + O(R^3) \end{aligned} \quad (3.36)$$

The crucial observation now is that the Riemann tensor of a weak euclidean field $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ is invariant with respect to gauge transformations. Thus we can substitute in (3.36) the Riemann tensor $R(\xi)$ in the radial gauge with that in the Feynman-De Witt gauge, say $R^F(x)$. To this end we also must express ξ in terms of x , but this will just affect the term with one single R . The function $x^\mu(\xi)$ is given by (3.13). Finally, eq. (3.36) can be rewritten as

$$\begin{aligned} e_\mu^a(\xi) = & \delta_\mu^a + \delta_\nu^a \xi^\alpha \xi^\beta \int_0^1 ds s(1-s) R_{\alpha\beta\mu}^{F\nu}(s\xi) + \\ & - \int_0^1 ds s^3(1-s) \int_0^1 dt (1-t) f_\mu^a(s, t, \xi) + o(R^3), \end{aligned} \quad (3.37)$$

where

$$f_\mu^a(s, t, \xi) = \delta_\nu^a \xi^\alpha \xi^\beta \xi^\rho \xi^\sigma \left[R_{\alpha\beta\mu,\gamma}^{F\nu}(x) \Gamma_{\rho\sigma}^{F\gamma}(y) - R_{\alpha\beta\gamma}^{F\nu}(x) R_{\rho\sigma\mu}^{F\gamma}(y) \right]_{x=s\xi, y=st\xi} \quad (3.38)$$

Eq. (3.37) describes the geodesic motion of the parallel-transported tetrad. Note that $e_\mu^a(\xi)$ still satisfies the gauge condition (3.7b). Let us now suppose that the gravitational

field consists of weak fluctuations quantized around a flat background. We can find the “average motion” of the tetrad by averaging (3.37) on the vacuum state. To this end we just need to replace the quantities in square brackets with their Feynman-De Witt propagators. The term linear in R will vanish in this approximation. One thus starts from the propagator of the metric in dimension 4 [De Witt, 1967 c; Veltman, 1976]

$$\langle h_{\mu\nu}^F(x) h_{\rho\sigma}^F(y) \rangle_0 = -\frac{\kappa^2}{(2\pi)^2} \frac{P_{\mu\nu\rho\sigma}}{(x-y)^2}, \quad (3.39)$$

where $\kappa = \sqrt{16\pi G}$ is the Planck length and

$$P_{\mu\nu\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma}. \quad (3.40)$$

Then one exploits the linearized expressions for the connection and the curvature

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2}\delta^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu}), \quad (3.41)$$

$$R_{\beta\mu\nu}^\alpha = \frac{1}{2}\delta^{\alpha\gamma}(\partial_\gamma\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\mu h_{\gamma\nu} - \partial_\gamma\partial_\nu h_{\beta\mu} + \partial_\beta\partial_\nu h_{\gamma\mu}) \quad (3.42)$$

and the formula

$$\xi^\alpha\xi^\beta\partial_\alpha\partial_\beta\partial_\mu\partial_\nu\frac{1}{\xi^2} = \frac{40}{\xi^6}(4\xi_\mu\xi_\nu - \xi^2\delta_{\mu\nu}). \quad (3.43)$$

In this way one finds, after a long but straightforward calculation

$$\langle f_\mu^a(s, t, \xi) \rangle_0 = \frac{160\ell^2}{(2\pi)^2} \frac{1}{s^7(1-t)^2} \frac{1}{\xi^2} \left(\delta_\mu^a - \frac{\xi^a\xi_\mu}{\xi^2} \right). \quad (3.44)$$

As we explained above, we must now take care of the finite dimension of the tetrad. It is apparent that (3.44) will give a divergent contribution after integration in s and t . This happens because the arguments of the fields in (3.38) coincide in some points. However, if we cut off the modes of the field with wavelenght smaller than L , the propagator (3.39) can be replaced by

$$\langle h^F(x) h^F(y) \rangle_0 \sim \frac{1}{(x-y)^2 + L^2}; \quad (3.45)$$

we implement this condition restricting the integration in s and t to a domain such that

$$|s\xi - st\xi| > L, \quad (3.46)$$

that is

$$s(1-t) > |\zeta|^{-1}, \quad \text{where } \zeta = \frac{\xi}{L} > 1.$$

Executing the integration we finally find

$$\langle e_\mu^a(\zeta) \rangle_0 = \delta_\mu^a + \frac{\ell^2}{L^2} \phi(|\zeta|) \left(\delta_\mu^a - \frac{\zeta^a \zeta_\mu}{\zeta^2} \right) + o(\ell^2), \quad (3.47)$$

where $\phi(|\zeta|)$ is an analytic function which vanishes at $\zeta = 1$ and whose form depends on the details of the regulator. Note that ζ expresses the distance in the new unit length L . In order our procedure to be meaningful, ζ must be not too large.

Equation (3.47) is the main result. It shows that the “physical” tetrads, parallel transported in the vacuum, differ from those of flat space by a correction proportional to the square of ℓ_{Planck}/L . Our position has been, of course, to consider the Feynman-De Witt gauge just as a way to fix the dynamical components of the field, whereas the true coordinates have to be constructed in an operational way. If one accepts this point of view, one concludes from eq. (3.47) that at very small distances, when L becomes comparable with ℓ , the size of the test particle influences its average motion in a fluctuating gravitational field.

4. RADIAL PROPAGATORS.

In this chapter we shall introduce a technique [Menotti and Seminara, 1991 b; Menotti, M. and Seminara, 1993] which allows to obtain the propagator of a gauge field in an arbitrary “sharp” gauge (that is, a gauge fixing obtained by insertion of a delta function in the functional integral), starting from the Feynman gauge. This technique, called the “projectors method” is very general, since it works in arbitrary dimension and can be easily specialized to electrodynamics, Yang-Mills theory and Einstein gravity in the first- or second-order formalism. It also allows to select the various solutions of the propagator equation, which are usually connected by a residual gauge transformation and have different regularity properties. In the case of the radial gauge, as we shall see, there are three propagators, which differ by the behaviour at the origin and at infinity. In Section 4.1 we shall explain the method in generic form, with emphasis on the algebraic properties; in the Section 4.2 we shall apply it to electrodynamics, specifying also in details for illustrative purposes the convergence and regularity properties, and in Section 4.3 we shall apply it to gravity. Finally, in Section 4.4, after comparing the radial gauge vacuum correlations with those of the Mandelstam covariant, we shall compute explicitly their components in a simplified case, in order to distinguish the components which are physically meaningful.

4.1 The gauge projectors method.

Let us consider a generic gauge field $A(x)$. A can be an electromagnetic field, or a Yang-Mills field, or finally the gravitational field, in the first or second order formalism. Let the linearized equation of motion for $A(x)$ in the presence of an external source be written in the form

$$K_x A(x) = -J(x),$$

where K is a linear, non-invertible, hermitean “kinetic” operator and $J(x)$ is an external source coupled to A . The gauge transformations of A have the form

$$A(x) \rightarrow A(x) + C_x f(x), \quad f(x) \text{ any function.}$$

Here C_x is another linear operator, which has the properties

$$K_x C_x = 0 \quad (\text{gauge invariance})$$

and

$$C_x^\dagger K_x = 0 \Rightarrow C_x^\dagger J(x) = 0 \quad (\text{source conservation}). \quad (4.1)$$

We assume that we can always add to the kinetic operator K an operator $K^{\mathcal{F}}$ which makes K invertible, as it happens, for example, in the Feynman gauge. We assume $K^{\mathcal{F}}$ to be of the form $\mathcal{F}^\dagger \mathcal{F}$, as deriving from a quadratic gauge fixing $\int dx \mathcal{F}^2(A(x))$. \mathcal{F} is

meant to be a linear operator on A . The propagator $G^{\mathcal{F}}$ corresponding to this gauge fixing satisfies

$$(K_x + K_x^{\mathcal{F}}) G^{\mathcal{F}}(x, y) = -\delta(x - y) \quad (4.2)$$

and has the following property

$$\int dy K_x^{\mathcal{F}} G^{\mathcal{F}}(x, y) J(y) = 0 \quad \text{if } C_y^{\dagger} J(y) = 0. \quad (4.3)$$

In other words, $K_x^{\mathcal{F}}$ vanishes when applied to the fields generated by physical sources. In fact, applying C^{\dagger} to (4.2) we get

$$C_x^{\dagger} K_x G^{\mathcal{F}}(x, y) + C_x^{\dagger} \mathcal{F}_x^{\dagger} \mathcal{F}_x G^{\mathcal{F}}(x, y) = C_x^{\dagger} \delta(x - y).$$

But using (4.1) we also have

$$C_x^{\dagger} \mathcal{F}_x^{\dagger} \mathcal{F}_x G^{\mathcal{F}}(x, y) = C_x^{\dagger} \delta(x - y)$$

and integrating on a conserved source J we obtain

$$\int dy C_x^{\dagger} \mathcal{F}_x^{\dagger} \mathcal{F}_x G^{\mathcal{F}}(x, y) J(y) = 0 \quad \text{if } C_y^{\dagger} J(y) = 0. \quad (4.4)$$

We notice that $\mathcal{F}C$ is the kinetic ghost operator and as such invertible. Then from (4.4) we get

$$\int dy \mathcal{F}_x G^{\mathcal{F}}(x, y) J(y) = 0$$

and multiplying by \mathcal{F}_x^{\dagger} we finally prove (4.3).

Let us now impose on A a generic “sharp” gauge condition \mathcal{G}

$$A^{\mathcal{G}}(x) = \{A(x) : \mathcal{G}(A(x)) = 0\}. \quad (4.5)$$

\mathcal{G} is meant to be a linear function of A and of its derivatives. The field $A^{\mathcal{G}}(x)$ can be obtained from a generic field $A(x)$ through a (generally non-local) projector $P^{\mathcal{G}}$

$$A^{\mathcal{G}}(x) = P^{\mathcal{G}}[A(x)] = A(x) + C_x F^{\mathcal{G}}[A(x)]. \quad (4.6)$$

We require this projector to be insensitive to any “previous gauge” of the field, namely to satisfy

$$P^{\mathcal{G}}[C_x f(x)] = 0, \quad \text{for any } f(x). \quad (4.7)$$

We shall now prove the following properties of the adjoint projector $P^{\mathcal{G}\dagger}$.

(1) $P^{\mathcal{G}^\dagger}$ produces conserved sources.

For, integrating (4.7) on a current $J(x)$ we have

$$\int dx P^{\mathcal{G}}[C_x f(x)] J(x) = 0;$$

by definition of the adjoint projector, this means that

$$\int dx (C_x f(x)) P^{\mathcal{G}^\dagger}[J(x)] = 0$$

and integrating by parts we have

$$\int dx f(x) C_x^\dagger P^{\mathcal{G}^\dagger}[J(x)] = 0$$

or, due to the arbitrariness of $f(x)$

$$C_x^\dagger P^{\mathcal{G}^\dagger}[J(x)] = 0 \quad \text{for any } J(x).$$

(2) $P^{\mathcal{G}^\dagger}$ leaves a conserved source unchanged.

Let us suppose that J is conserved. Using (4.6) we have for a generic A

$$\int dx A(x) P^{\mathcal{G}^\dagger}[J(x)] = \int dx A(x) J(x) + \int dx \{C_x F^{\mathcal{G}}[A(x)]\} J(x).$$

Integrating by parts the second term on the r.h.s. and using (4.1) and the arbitrariness of A we have

$$P^{\mathcal{G}^\dagger}[J(x)] = J(x) \quad \text{if } C_x^\dagger J(x) = 0. \quad \text{q.e.d.}$$

The equation of motion obtained varying the action under the constraint (4.5) is

$$P^{\mathcal{G}^\dagger}[K_x A^{\mathcal{G}}(x)] = -P^{\mathcal{G}^\dagger}[J(x)].$$

From (4.1) and Property 2 we have

$$K_x A^{\mathcal{G}}(x) = -P^{\mathcal{G}^\dagger}[J(x)],$$

or for the propagator

$$K_x G^{\mathcal{G}}(x, y) = -P^{\mathcal{G}^\dagger}[\delta(x - y)], \quad (4.8)$$

where the meaning of the r.h.s. is

$$\int dy P^{\mathcal{G}^\dagger}[\delta(x - y)] J(y) = P^{\mathcal{G}^\dagger}[J(x)].$$

Next we show that a solution of (4.8) is

$$G^{\mathcal{G}}(x, y) = \langle P^{\mathcal{G}}[A^{\mathcal{F}}(x)] P^{\mathcal{G}}[A^{\mathcal{F}}(y)] \rangle_0, \quad (4.9)$$

where $A^{\mathcal{F}}$ denotes the field in the original gauge. Integrating on a source $J(y)$ we have

$$\begin{aligned} & \int dy K_x \langle P^{\mathcal{G}}[A^{\mathcal{F}}(x)] P^{\mathcal{G}}[A^{\mathcal{F}}(y)] \rangle_0 J(y) = \\ &= \int dy K_x \langle P^{\mathcal{G}}[A^{\mathcal{F}}(x)] A^{\mathcal{F}}(y) \rangle_0 P^{\mathcal{G}}[J(y)] = \\ &= \int dy K_x \langle \{A^{\mathcal{F}}(x) + C_x F^{\mathcal{G}}[A^{\mathcal{F}}(x)]\} A^{\mathcal{F}}(y) \rangle_0 P^{\mathcal{G}}[J(y)] = \\ &= \int dy K_x \langle A^{\mathcal{F}}(x) A^{\mathcal{F}}(y) \rangle_0 P^{\mathcal{G}}[J(y)] = \\ &= \int dy (K_x + K_x^{\mathcal{F}}) G^{\mathcal{F}}(x, y) P^{\mathcal{G}}[J(y)] - \int dy K_x^{\mathcal{F}} G^{\mathcal{F}}(x, y) P^{\mathcal{G}}[J(y)] = \\ &= - \int dy \delta(x - y) P^{\mathcal{G}}[J(y)]. \end{aligned} \quad (4.10)$$

In the last step we have used (4.3) and Property 1. Note however that $G^{\mathcal{G}}$ defined in (4.9) remains unchanged if we replace the original field with any other gauge equivalent field.

Given two different projectors P_1 and P_2 which project on the same gauge (which as a rule differ for different boundary conditions), one has

$$P_1 P_2 = P_1; \quad P_2 P_1 = P_2, \quad (4.11)$$

due to (4.7). Thus also $P_{12} = \alpha P_1 + (1 - \alpha) P_2$ is a projector on the considered gauge and one can write down the P_{12} -projected Green function equation (we omit the suffix \mathcal{G})

$$K_x G(x, y) = P_{12}^{\dagger}[\delta(x - y)]. \quad (4.12)$$

It is immediate to verify that a solution of (4.12) is also given by

$$\alpha \langle P_2[A(x)] P_1[A(y)] \rangle_0 + (1 - \alpha) \langle P_1[A(x)] P_2[A(y)] \rangle_0. \quad (4.13)$$

Namely, repeating the same procedure of eq. (4.10) we have

$$K_x \alpha \langle P_2[A(x)] P_1[A(y)] \rangle_0 = \alpha K_x \langle A(x) P_1[A(y)] \rangle_0 = \alpha P_1^{\dagger}[\delta(x - y)].$$

Acting similarly with the $(1 - \alpha)$ term in (4.13), we get (4.12). We notice that for $\alpha = \frac{1}{2}$, (4.13) is symmetric in the exchange of the field arguments.

We close this section writing in explicit form the operators which appear in electrodynamics and linearized Einstein theory.

Electrodynamics and linearized Yang-Mills theory.

This is the most simple case. We have the following identifications

$$\begin{aligned} A &\rightarrow A_\mu; \\ J &\rightarrow J_\mu; \\ Cf &\rightarrow \partial_\mu f; \\ K &\rightarrow \delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu; \\ K^{\mathcal{F}} &\rightarrow \partial_\mu \partial_\nu, \end{aligned}$$

where $K^{\mathcal{F}}$ is the operator which is produced by the usual Feynman gauge fixing $\frac{1}{2}(\partial^\mu A_\mu)^2$.

Linearized Einstein gravity in the second-order formalism.

In this case we have

$$\begin{aligned} A &\rightarrow h_{\mu\nu}; \\ J &\rightarrow T_{\mu\nu}; \\ Cf &\rightarrow (\delta_{\alpha\sigma} \partial_\rho + \delta_{\alpha\rho} \partial_\sigma) f_\alpha; \\ K &\rightarrow K_{\mu\nu\rho\sigma} = \frac{1}{4} [2 \delta_{\rho\sigma} \partial_\mu \partial_\nu + 2 \delta_{\mu\nu} \partial_\rho \partial_\sigma + \\ &\quad - (\delta_{\mu\rho} \partial_\nu \partial_\sigma + \delta_{\nu\rho} \partial_\mu \partial_\sigma + \delta_{\mu\sigma} \partial_\nu \partial_\rho + \delta_{\nu\sigma} \partial_\mu \partial_\rho) + \\ &\quad + (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} - 2 \delta_{\mu\nu} \delta_{\rho\sigma}) \partial^2]; \\ K^{\mathcal{F}} &\rightarrow K_{\mu\nu\rho\sigma}^{\mathcal{F}} = \frac{1}{4} [-(2 \delta_{\rho\sigma} \partial_\mu \partial_\nu + 2 \delta_{\mu\nu} \partial_\rho \partial_\sigma) + \\ &\quad + (\delta_{\mu\rho} \partial_\nu \partial_\sigma + \delta_{\nu\rho} \partial_\mu \partial_\sigma + \delta_{\mu\sigma} \partial_\nu \partial_\rho + \delta_{\nu\sigma} \partial_\mu \partial_\rho)]. \end{aligned}$$

Here $K^{\mathcal{F}}$ is the operator produced by the harmonic gauge fixing

$$\frac{1}{2} \left(\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial^\nu h_\mu^\mu \right)^2. \quad (4.14)$$

Linearized Einstein gravity in the first-order formalism.

The quadratic part of the lagrangian has the form

$$L^{(2)} = -(\delta_{abc}^{\mu\nu\gamma} \partial_\mu \Gamma_\nu^{ab} \tau_\gamma^c + \delta_{ab}^{\mu\nu} \Gamma_{c\mu}^a \Gamma_\nu^{cb} + T_a^\mu \tau_\mu^a + \Sigma_{ab}^\mu \Gamma_\mu^{ab}),$$

where

$$\begin{aligned} \tau_\mu^a &= e_\mu^a - \delta_\mu^a \\ \delta_{ab}^{\mu\nu} &= \delta_a^\mu \delta_b^\nu - \delta_b^\mu \delta_a^\nu \\ \delta_{abc}^{\mu\nu\gamma} &= \delta_a^\mu \delta_{bc}^{\nu\gamma} - \delta_b^\mu \delta_{ac}^{\nu\gamma} + \delta_c^\mu \delta_{ab}^{\nu\gamma} \end{aligned}$$

and T_a^μ and Σ_{ab}^μ are the energy-momentum source and the spin-torsion source, respectively. The gauge transformations have the form

$$Cf \rightarrow \begin{pmatrix} 0 & \delta_{cd}^{ab} \partial_\mu \\ \partial_\mu & -\delta_{\mu b} \delta_{cd}^{ab} \end{pmatrix} \begin{pmatrix} \Lambda^a \\ \theta^{cd} \end{pmatrix}.$$

The field equations are given by

$$KA = J \rightarrow \begin{pmatrix} \frac{1}{2}(\delta_{c\sigma} \delta_{dab}^{\nu\sigma\mu} - \delta_{d\sigma} \delta_{cab}^{\nu\sigma\mu}) & -\delta_{rab}^{\lambda\gamma\mu} \partial_\lambda \\ -\delta_{abc}^{\mu\lambda\nu} \partial_\mu & 0 \end{pmatrix} \begin{pmatrix} \Gamma_\nu^{cd} \\ \tau_\gamma^r \end{pmatrix} = \begin{pmatrix} \Sigma_{ab}^\mu \\ T_d^\mu \end{pmatrix}$$

and the gauge-fixing term has the form

$$K^F A \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 4K_{a\mu b\nu}^F \delta^{\nu\gamma} + 2\beta (\delta_{ab} \delta_\mu^\gamma - \delta_{\mu b} \delta_a^\gamma) \end{pmatrix} \begin{pmatrix} \Gamma_\nu^{ab} \\ \tau_\gamma^b \end{pmatrix},$$

and is produced by the harmonic gauge fixing (4.14) with $h_{\mu\nu} = \tau_\mu^a \delta_{a\nu} + \tau_\nu^a \delta_{a\mu}$, to which the symmetric gauge fixing $\frac{\beta}{2}(\tau_\mu^a \delta_{a\nu} - \tau_\nu^a \delta_{a\mu})^2$ has been added.

4.2 Electrodynamics. Regularity properties.

In this Section we shall apply the projectors method to the case of electrodynamics or linearized Yang-Mills theory in radial gauge [Menotti and Seminara, 1991 b]. We start from the inversion formula (3.2), writing it in the form

$$A_\mu^0(x) = P^0[A]_\mu(x) = x_\rho \int_0^1 d\lambda \lambda F_{\rho\mu}(\lambda x). \quad (4.15)$$

Let be $r = |x|$. The integral in (4.15) converges if $|F(x)| < r^{-2+\varepsilon}$ as $r \rightarrow 0$. We notice that if the field $A_\mu(x)$ is such that $|A(x)| < r^{-1+\varepsilon}$ for $r \rightarrow 0$, then (4.15) can be rewritten as

$$A_\mu^0(x) = P^0[A]_\mu(x) = A_\mu(x) - \frac{\partial}{\partial x_\mu} \int_0^1 d\lambda x_\rho A_\rho(\lambda x). \quad (4.16)$$

This shows that the general form (4.6) of the propagator is maintained, provided the fields are not too much singular. There is, however, a limiting case in which (4.16) is not true any more: namely, when we start from a field which is already radial and differs from $A_\mu^0(x)$ by a residual gauge transformation of the form (3.1). For instance, let us take as starting field the radial field $A_\mu(x) = -x_\rho \int_1^\infty d\lambda \lambda F_{\rho\mu}(\lambda x)$. Suppose that $F_{\mu\nu}(x)$ is finite at the origin. It is easy to see that $|A(x)| \sim r^{-1}$ as $r \rightarrow 0$, while F keeps finite. This happens because the singular part of A is a pure gauge, which does not contributes to

F. The exact significance of this detail will become more clear as we proceed with our discussion. In practice, we assume the following definition of P^0 .

Definition of P^0 :

$$A_\mu^0(x) \equiv P^0[A]_\mu(x) \equiv x_\rho \int_0^1 d\lambda \lambda F_{\rho\mu}(\lambda x). \quad (4.17)$$

In a similar way, we can define a projector in which the integration limits 0 and ∞ are exchanged.

Definition of P^∞ :

$$A_\mu^\infty(x) \equiv P^\infty[A]_\mu(x) \equiv -x_\rho \int_1^\infty d\lambda \lambda F_{\rho\mu}(\lambda x). \quad (4.18)$$

It is easy to verify that the difference between A^0 and A^∞ is a residual gauge term of the form (3.1).

It is apparent from (4.17), (4.18) that P^0 and P^∞ are projection operators, namely we have $(P^0)^2 = P^0$, $(P^\infty)^2 = P^\infty$. In addition, the following properties hold (compare (4.11))

$$P^0 P^\infty = P^0; \quad P^\infty P^0 = P^\infty.$$

The definition domains of P^0 and P^∞ are respectively given by $\{F : |F(x)| < r^{-2+\varepsilon} \text{ for } r \rightarrow 0\}$ and $\{F : |F(x)| < r^{-2-\varepsilon} \text{ for } r \rightarrow \infty\}$. Restricting such domains respectively to

$\{F : |F(x)| < r^{-1+\varepsilon} \text{ for } r \rightarrow 0\}$ and $\{F : |F(x)| < r^{-3-\varepsilon} \text{ for } r \rightarrow \infty\}$,

we find that the fields A^0 and A^∞ can be characterized in a simple way through their asymptotic behaviour. Namely, we find that

- $A_\mu^0(x)$ vanishes at the origin and decreases like r^{-1} at infinity;
- $A_\mu^\infty(x)$ grows like r^{-1} at the origin and vanishes like $r^{-1-\varepsilon}$ at infinity.

In fact, let \hat{x} be the unit verson of x . We have

$$A_\mu^0(x) = x_\rho \int_0^1 d\lambda \lambda F_{\rho\mu}(\lambda r \hat{x}) = \frac{x_\rho}{r^2} \int_0^r dt t F_{\rho\mu}(t \hat{x}). \quad (4.19)$$

In the limit $r \rightarrow 0$ we obtain (in the domain above)

$$A^0(x) < \frac{1}{r} [t^{1+\varepsilon}]_0^r \rightarrow 0 \quad \text{for } r \rightarrow 0.$$

On the other hand, in the limit $r \rightarrow \infty$, the integral converges and A^0 has the form

$$A_\mu^0(x) = \frac{1}{r} H_\mu^0(x) \quad \text{for } r \rightarrow \infty,$$

where $H_\mu^0(x)$ is a homogeneous function of degree 0. Analogous formulas hold for A^∞ . In the limit $r \rightarrow 0$, A^∞ is regular, in the sense that it vanishes faster than r^{-1} . At the origin we have instead

$$A_\mu^\infty(x) = -\frac{1}{r} H_\mu^0(x) \quad \text{for } r \rightarrow 0. \quad (4.20)$$

From eq.s (4.19) - (4.20) we see that if we compute a radial Wilson loop (fig. 1) going to infinity, we obtain the same result both using A^0 and A^∞ . In the first case the effective contribution is localized at infinity, in the second case at the origin. Finally, if we define the mixed projector

$$P^S = \frac{1}{2}(P^0 + P^\infty),$$

we find that the contribution of the field A^S to the radial loop is splitted into two parts: one half at the origin, one half at infinity.

For the computation of the adjoint projectors $P^{0\dagger}$, $P^{\infty\dagger}$ and $P^{S\dagger}$ we refer to the mentioned work of Menotti and Seminara.

By means of the projectors P^0, P^∞, P^S we can now write down three symmetric radial propagators, denoted by G^0, G^∞ and G^S :

$$\begin{aligned} G_{\mu\nu}^0(x, y) &= \langle P^0[A]_\mu(x) P^0[A]_\nu(y) \rangle_0 = \\ &= x^\rho y^\sigma \int_0^1 d\lambda \lambda \int_0^1 d\tau \tau \langle F_{\rho\mu}(\lambda x) F_{\sigma\nu}(\tau y) \rangle_0; \end{aligned}$$

$$\begin{aligned} G_{\mu\nu}^\infty(x, y) &= \langle P^\infty[A]_\mu(x) P^\infty[A]_\nu(y) \rangle_0 = \\ &= x^\rho y^\sigma \int_1^\infty d\lambda \lambda \int_1^\infty d\tau \tau \langle F_{\rho\mu}(\lambda x) F_{\sigma\nu}(\tau y) \rangle_0; \end{aligned}$$

$$\begin{aligned} G_{\mu\nu}^S(x, y) &= \frac{1}{2} \langle P^0[A]_\mu(x) P^\infty[A]_\nu(y) \rangle_0 + \frac{1}{2} \langle P^\infty[A]_\mu(x) P^0[A]_\nu(y) \rangle_0 = \\ &= -\frac{1}{2} x^\rho y^\sigma \int_1^\infty d\lambda \lambda \int_0^1 d\tau \tau \langle F_{\rho\mu}(\lambda x) F_{\sigma\nu}(\tau y) \rangle_0 \\ &\quad - \frac{1}{2} x^\rho y^\sigma \int_0^1 d\lambda \lambda \int_1^\infty d\tau \tau \langle F_{\rho\mu}(\lambda x) F_{\sigma\nu}(\tau y) \rangle_0. \end{aligned}$$

To compute them explicitly we make use of the formula

$$x^\rho y^\sigma \langle F_{\rho\mu}(x) F_{\sigma\nu}(y) \rangle_0 = x^\rho y^\sigma D_{\mu\rho\nu\sigma} \mathcal{D}(x - y),$$

where

$$D_{\mu\rho\nu\sigma} = \partial_{\mu\rho} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial y^\sigma} + \partial_{\nu\sigma} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\rho} - \partial_{\mu\sigma} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial y^\rho} - \partial_{\nu\rho} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\sigma}$$

and $\mathcal{D}(x-y)$ is the Feynman propagator. It can be shown that the integrals of G^0 converge in dimension $N < 4$, while those of G^∞ converge for $N > 4$; finally, the integrals of G^S converge for $N > 3$. Let us compute explicitly, for instance, G^S . Using the formula

$$\int_0^1 d\rho \frac{1}{[(\rho x - y)^2]^{\frac{N}{2}-1}} = \int_0^\infty d\rho \frac{1}{[(\rho x - y)^2]^{\frac{N}{2}-1}} - \int_0^1 d\rho \frac{\rho^{N-4}}{[(x - \rho y)^2]^{\frac{N}{2}-1}}$$

we can write the integral of interest in the form

$$D(x, y) = -\frac{\Gamma(\frac{N}{2} - 1)}{4\pi^{N/2}} \frac{1}{N-4} \left\{ \int_0^1 d\rho \rho^{N-4} \left[\frac{1}{[(\rho x - y)^2]^{\frac{N}{2}-1}} + \frac{1}{[(x - \rho y)^2]^{\frac{N}{2}-1}} \right] - \int_0^\infty d\rho \frac{1}{[(\rho x - y)^2]^{\frac{N}{2}-1}} \right\}. \quad (4.21)$$

The first integral which appears in (4.21) can be rewritten as follows

$$I_1 = \int_0^1 d\rho \frac{\rho^{N-4}}{[\rho^2 X^2 - 2\rho X Y \cos \theta + Y^2]^{N/2-1}},$$

where $X = |x|$, $Y = |y|$, and performing the change of variables $\rho = \frac{Y\zeta}{\sqrt{d}\zeta + 1}$, with $d = \sqrt{X^2 + Y^2 - 2XY \cos \theta}$, we obtain [Erdélyi et al., 1953]

$$\begin{aligned} & Y^{-1} d^{3-N} \int_0^\infty d\zeta \frac{\zeta^{N-4}}{\left[\zeta^2 + 2\zeta \left(\frac{Y-X \cos \theta}{d} \right) + 1 \right]^{N/2-1}} = \\ & = \frac{1}{N-3} Y^{-1} d^{3-N} (4 \sin^2 \phi_x)^{\frac{3-N}{4}} \Gamma \left(\frac{N-1}{2} \right) P_{\frac{3-N}{2}}^{\frac{3-N}{2}} (\cos \phi_x), \end{aligned}$$

being $\cos \phi_x = \frac{Y-X \cos \theta}{d}$. Here θ is the angle between x and y , and ϕ_x is the angle opposite to x in the euclidean triangle of sides x , y , d . Expressing $P_{\frac{3-N}{2}}^{\frac{3-N}{2}}$ in terms of hypergeometric functions, we have

$$I_1 = \frac{1}{N-3} \frac{d^{3-N}}{Y^{-1}} \left(\cos^2 \frac{\phi_x}{2} \right)^{\frac{3-N}{2}} {}_2F_1 \left(\frac{N-3}{2}, \frac{5-N}{2}; \frac{N-1}{2}; \sin^2 \frac{\phi_x}{2} \right).$$

In a similar way we can compute

$$I_2 = \int_0^\infty d\rho \frac{1}{[\rho^2 X^2 - 2\rho X Y \cos \theta + Y^2]^{N/2-1}},$$

obtaining

$$I_2 = \frac{1}{N-3} \frac{Y^{3-N}}{X^{-1}} \left(\sin^2 \frac{\theta}{2} \right)^{\frac{3-N}{2}} {}_2F_1 \left(\frac{N-3}{2}, \frac{5-N}{2}; \frac{N-1}{2}; \cos^2 \frac{\theta}{2} \right).$$

The behaviour of $[D(x, y) + D(y, x)]$ as $X \rightarrow 0$, and y and θ are constant, is obtained from I_1 and I_2 by noticing that, in this limit, $\phi_x \rightarrow 0$ and $\phi_y \rightarrow \pi - \theta$. We then have

$$\begin{aligned} D(x, y) + D(y, x) &\simeq \\ &\simeq \frac{1}{N-3} \left(\sin^2 \frac{\theta}{2} \right)^{\frac{3-N}{2}} {}_2F_1 \left(\frac{N-3}{2}, \frac{5-N}{2}; \frac{N-1}{2}; \cos^2 \frac{\theta}{2} \right) \times \\ &\quad \left(\frac{Y^{3-N}}{X^{-1}} - \frac{X^{3-N}}{Y^{-1}} \right). \end{aligned}$$

This is also the behaviour for x and θ fixed, while $Y \rightarrow \infty$. Owing to the symmetry of $[D(x, y) + D(y, x)]$, we have that for $Y \rightarrow 0$, x and θ fixed

$$\begin{aligned} D(x, y) + D(y, x) &\simeq \\ &\simeq -\frac{1}{N-3} \left(\sin^2 \frac{\theta}{2} \right)^{\frac{3-N}{2}} {}_2F_1 \left(\frac{N-3}{2}, \frac{5-N}{2}; \frac{N-1}{2}; \cos^2 \frac{\theta}{2} \right) \times \\ &\quad \left(\frac{Y^{3-N}}{X^{-1}} - \frac{X^{3-N}}{Y^{-1}} \right) \end{aligned}$$

and the same holds for $X \rightarrow \infty$, y and θ fixed. Hence we see that the behaviour for $X \rightarrow \infty$ and $X \rightarrow 0$ is the same, except for the sign. Being $D_{\mu\nu\rho\sigma}$ a zero degree operator, this also holds for the whole propagator $G_{\mu\nu}^S(x, y)$.

4.3 Gravitational propagators.

In Section 4.1 we have developed in general form the projectors method, which provides an algebraic way for constructing the propagator in any “sharp” gauge, provided it is known in a Feynman-like gauge. Then in Section 4.2 the projectors and propagators of electrodynamics were explicitly written, taking care of their existence and regularity properties.

The reason for which it is desirable to impose the radial gauge in gravity as a sharp gauge (that is, to insert a delta into the functional integral) is that the geometrical meaning of the radial gauge can be formally preserved, at the quantum level, only if the fields admitted in the functional integration satisfy the gauge condition.

The functional integral of gravity in the first order formalism is given by†

$$z = \int d[\Gamma_\mu^{ab}] d[e_\mu^a] \mu[e_\mu^a] e^{-\ell^{-2} \int d^N x R^{ab} \wedge e^c \dots \varepsilon_{abc} \dots}.$$

The gauge fixing term is $\delta(x^\mu \Gamma_\mu^{ab}) \delta(x^\mu (e_\mu^a - \delta_\mu^a))$. Note that it preserves the symmetry between vierbein and connection.

† It is well known that the rigorous definition of these integrals is affected by the unboundedness of the euclidean action and by the arbitrariness of the functional integration measure [see Menotti, 1990]. We shall not dwell on such problems here.

As it happens in Yang-Mills theory, the ghosts associated to the local Lorentz symmetry formally decouple, while those associated to the diffeomorphisms survive (in the first- as well as in the second order formalism).

We remind that in the functional integral approach the correlation functions are computed by averaging the products of fields like $O(x)O(y)$ on all geometries weighted by the exponential of the gravitational action. Fixing the gauge to the radial gauge gives x, y, \dots a well defined meaning, as the points that acquire geodesic coordinates x, y, \dots in each of the geometries we are summing over. Hence, when the points lie on a straight line through the origin, their correlation is automatically geodesic.

Next we write the expressions for the radial projectors which are the analogue of the P^0 , P^∞ and P^S of the preceding Section.

P^0 is given by (3.17) and (3.19) substituting to $R_{\mu\nu}^{ab}$ and $S_{\mu\nu}^a$ their linearized expressions

$$\begin{aligned} R_{\mu\nu}^{L,ab}(x) &= \partial_\mu \Gamma_\nu^{ab}(x) - \partial_\nu \Gamma_\mu^{ab}(x), \\ S_{\mu\nu}^{L,a}(x) &= \partial_\mu \tau_\nu^a(x) - \partial_\nu \tau_\mu^a(x) + \Gamma_\mu^{ab}(x) \delta_{b\nu} - \Gamma_\nu^{ab}(x) \delta_{b\mu}, \end{aligned}$$

which are invariant under linearized gauge transformations. We thus have

$$P^0 \begin{pmatrix} \Gamma_{b,\mu}^a(x) \\ \tau_\mu^a(x) \end{pmatrix} = \begin{pmatrix} x^\nu \int_0^1 d\lambda \lambda R_{b\nu\mu}^{La}(\lambda x) \\ x^\nu x^b \int_0^1 d\lambda \lambda (1-\lambda) R_{b\nu\mu}^{La}(\lambda x) + x^\nu \int_0^1 d\lambda \lambda S_{\nu\mu}^{La}(\lambda x) \end{pmatrix} \quad (4.22)$$

$$P^\infty \begin{pmatrix} \Gamma_{b,\mu}^a(x) \\ \tau_\mu^a(x) \end{pmatrix} = \begin{pmatrix} -x^\nu \int_1^\infty d\lambda \lambda R_{b\nu\mu}^{La}(\lambda x) \\ -x^\nu x^b \int_1^\infty d\lambda \lambda (1-\lambda) R_{b\nu\mu}^{La}(\lambda x) - x^\nu \int_1^\infty d\lambda \lambda S_{\nu\mu}^{La}(\lambda x) \end{pmatrix} \quad (4.23)$$

The projector P^0 produces radial fields which are regular at the origin (i.e. behaving like the original fields) and gives a connection $\Gamma_\mu^{ab}(x)$ behaving like $1/r$ at infinity. On the other hand P^∞ projects on a radial field which is regular at infinity and gives a connection behaving like $1/r$ at the origin.

P^S , which allows us to construct a finite propagator in $N > 2$, treats the origin and infinity in symmetrical way by giving the same $1/r$ behavior in the two limits with opposite coefficients. This can be shown in a similar way as we did for electrodynamics [Menotti, M. and Seminara, 1993].

The adjoint projectors can also be easily computed and their expressions can be found in [Menotti, M. and Seminara, 1993], like many other details that we shall omit in the following.

When we construct by these projectors the propagators

$$\langle P^0(\Gamma, \tau) P^0(\Gamma, \tau) \rangle_0$$

and

$$\langle P^\infty(\Gamma, \tau) P^\infty(\Gamma, \tau) \rangle_0$$

we find that the first one is always divergent, while the second one diverges for $N \leq 4$. Thus we shall construct the solution for the P^S projected Green's function equation, namely

$$\begin{aligned} & \begin{pmatrix} \frac{1}{2}(\delta_{am}^{\mu\nu}\delta_{bn} - \delta_{an}^{\mu\nu}\delta_{bm}) & \delta_{abd}^{\lambda\mu\sigma}\partial_\lambda \\ \delta_{p mn}^{\mu\lambda\nu}\partial_\lambda & 0 \end{pmatrix} \begin{pmatrix} G_{\nu,\gamma}^{mn,rs} & G_{\nu,\beta}^{mn,g} \\ G_{\sigma,\gamma}^{d,rs} & G_{\sigma,\beta}^{d,g} \end{pmatrix} = \\ & = -P^{S\dagger} \begin{pmatrix} \delta_\gamma^\mu\delta_{ab}^{rs} & 0 \\ 0 & \delta_\beta^\mu\delta_p^g \end{pmatrix} \delta^N(x - y). \end{aligned} \quad (4.24)$$

by use of the now familiar procedure. One shows that the propagator

$$\frac{1}{2} \left\langle P^0 \begin{pmatrix} \Gamma_{b,\mu}^a(x) \\ \tau_\mu^a(x) \end{pmatrix} P^\infty(\Gamma_{b',\mu'}^{a'}(y), \tau_{\mu'}^{a'}(y)) \right\rangle + (P^0 \leftrightarrow P^\infty) \quad (4.25)$$

is a convergent symmetric radial solution of (4.24) for all $N > 2$. The explicit form of solution (4.25) of the Green's function equation (4.24) is easily computed by using (4.22) and (4.23) where the correlators between Riemann and torsion two-forms, which are invariant under linearized gauge transformations, can be obtained using e.g. the usual symmetric harmonic gauge.

Let be $M_{\mu,\nu}^{ab,cd}(x, y)$ the ultra-local part of the propagator of the anholonomic connection (that is, the part which is proportional to $\delta^N(x - y)$). M is given by the expression [Menotti and Pelissetto, 1987]

$$M_{\mu,\nu}^{ab,cd}(x, y) = -\frac{i}{4} \left(\delta_{\mu\nu} \delta^{ab,cd} + \delta_{\mu\gamma}^{ab} \delta_{\nu\gamma}^{cd} - \frac{2}{N-2} \delta_{\nu\gamma}^{ab} \delta_{\mu\gamma}^{cd} \right) \delta^N(x - y).$$

($\delta^{ab,cd}$ is an antisymmetric symbol in the pairs (a, b) , (c, d) .)

We have

$$\langle R_{\mu\nu}^{L,ab}(x) R_{\rho\sigma}^{L,cd}(y) \rangle = \langle R_{\mu\nu}^{L,ab}(x) R_{\lambda\rho}^{L,cd}(y) \rangle^{II} + [\partial_\mu \partial_\rho M_{\nu,\sigma}^{ab,cd}(x, y)]_{[\mu\nu],[\rho\sigma]}; \quad (4.26)$$

$$\langle S_{\mu\nu}^{L,a}(x) S_{\rho\sigma}^{L,b}(y) \rangle = [M_{\mu,\rho}^{ac,bd}(x, y) \delta_{c\nu} \delta_{d\sigma}]_{[\mu\nu],[\rho\sigma]}; \quad (4.27)$$

$$\langle R_{\mu\nu}^{L,ab}(x) S_{\rho\sigma}^{L,c}(y) \rangle = [\partial_\mu M_{\nu,\rho}^{ab,cd}(x, y) \delta_{d\sigma}]_{[\mu\nu],[\rho\sigma]}. \quad (4.28)$$

In (4.26), $\langle \rangle^{II}$ denotes the correlator in the second order formalism:

$$\begin{aligned} \langle R_{\mu\nu\alpha\beta}(x) R_{\rho\sigma\lambda\gamma}(y) \rangle_0 &= \frac{1}{4} \delta_{\mu\nu}^{\mu'\nu'} \times \\ &\quad \left(\delta_{\rho\sigma}^{\rho'\nu'} \delta_{\alpha\beta}^{\alpha'\beta'} \delta_{\lambda\gamma}^{\lambda'\beta'} + \delta_{\lambda\gamma}^{\lambda'\nu'} \delta_{\alpha\beta}^{\alpha'\beta'} \delta_{\rho\sigma}^{\rho'\beta'} - \frac{2}{N-2} \delta_{\alpha\beta}^{\alpha'\nu'} \delta_{\rho\sigma}^{\rho'\sigma'} \delta_{\lambda\gamma}^{\lambda'\sigma'} \right) \times \\ &\quad \partial_{\mu'} \partial_{\alpha'} \partial_{\rho'} \partial_{\lambda'} \mathcal{D}(x-y). \end{aligned}$$

The ultra-local nature of the correlators (4.27) and (4.28) reflects the well known fact that the torsion does not propagate in the Einstein-Cartan theory.

The integrals over λ and τ of the $\langle RR \rangle^{II}$ correlators are similar to those given for electrodynamics (Section 4.2). The contact terms generated by M give rise to integrals in λ and τ which are convergent. In fact the generic integral is of the form

$$\begin{aligned} &\int_0^1 d\lambda \lambda^A \int_1^\infty d\tau \tau^B \delta^N(\lambda x - \tau y) = \\ &= \int_0^1 d\lambda \lambda^A \int_1^\infty d\tau \tau^B \delta(\lambda|x| - \tau|y|) (\lambda|x|)^{-N+1} \delta^{N-1}(\Omega_x - \Omega_y) = \\ &= \delta^{N-1}(\Omega_x - \Omega_y) \Theta(|x| - |y|) \frac{1}{A+B-N+2} \times \\ &\quad (|y|^{-B-1} |x|^{B-N+1} - |y|^{A-N+1} |x|^{-A-1}). \end{aligned}$$

We notice that in three dimensions the correlator (4.26) is identically zero, as can be explicitly verified by observing that the matrix component (2,1) of eq. (4.24) takes the form $\langle R^L(x)\Gamma(y) \rangle \equiv 0$, which implies $\langle R^L(x)R^L(y) \rangle \equiv 0$. This means in turn, through (4.22) and (4.23), that $\langle \Gamma(x)\Gamma(y) \rangle \equiv 0$. Hence $\langle \tau(x)\tau(y) \rangle$ vanishes, except for singular contributions which arise when the origin, x and y are collinear; such contributions are given by (4.26), (4.27) and (4.28), that is, by the torsion-Riemann and torsion-torsion correlators.

In the second order formalism, the Riemann-Riemann correlator, which in Minkowski space corresponds to the T^* product, is not identically zero even in dimension $N = 3$. Namely, if this were the case, *all* propagators would vanish identically; but this cannot be true, since the collinear singularity must be present in the propagator $\langle h(x)h(y) \rangle$ in order to produce the conical defect which is typical of 3D gravity.

A merit of the radial gauge is to show explicitly the absence of propagation in the three-dimensional theory. The harmonic gauge – on the contrary – propagates a pure gauge field.

4.4 Relation to the Mandelstam covariant. Significant components.

We recall that the “Mandelstam covariant” $\mathcal{R}_{abcd}(x, P)$ (Section 2.1) is the curvature tensor observed in the local reference frame parallel transported to a point from a fixed origin x along the path P . In the version of Tsamis and Woodard (Section 2.2) the path P is assumed to be geodesic.

It is then clear that in the radial gauge the (geodesic) Mandelstam covariant referred to the origin of the coordinates can be written as

$$\mathcal{R}_{abcd}(\xi) = e_c^\mu(\xi) e_d^\nu(\xi) R_{ab\mu\nu}(\xi),$$

where $R_{ab\mu\nu}$ is given by eq. (3.15). One verifies from this expression that \mathcal{R} is a scalar under coordinates transformations.

Thus the (geodesic) Mandelstam covariant is a composite field which can be constructed by radial fields. Its correlations can be written using the propagators of $e_a^\mu(\xi)$ and $\Gamma_{\mu b}^a(\xi)$ given in Section 4.3, and the resulting analytical structure will be the same, including hypergeometric functions etc. The index structure is quite complicated however, and it is more interesting instead to show explicitly the components of the radial vierbein-vierbein correlation function, in order to gain a feeling of its behavior.

Denoting as usual the vierbein by $e_{a\mu} = \delta_{a\mu} + \tau_{a\mu}$ and all the integration procedure of the propagator (see Section 4.3) by $\int d[\lambda] \int d[t]$, we may write

$$\langle \tau_{a\mu}(x) \tau_{c\rho}(y) \rangle = \int d[\lambda] \int d[t] [x^\nu x^b y^\sigma y^d \langle R_{ab\mu\nu}(x) R_{cd\rho\sigma}(y) \rangle]_{x=\lambda x; y=ty}. \quad (4.29)$$

In this formula, $R_{ab\mu\nu}(x)$ denotes the usual ‘‘holonomic’’ components of the (linearized) Riemann tensor, that is, $R_{ab\mu\nu}(x) = \delta_a^\alpha \delta_b^\beta R_{\alpha\beta\mu\nu}(x)$, and similarly for $R_{cd\rho\sigma}$. Also, ‘‘ultralocal’’ correlations are disregarded (compare eq. (4.26)).

According to our usual assumptions, the two vierbein fields have to be connected by a geodesic of length ξ ; this can be realized for instance by choosing the radial coordinates x and y in the following way:

$$\begin{aligned} x_0 &= \left(-\frac{\xi}{2}, 0, 0, 0 \right); \\ y_0 &= \left(\frac{\xi}{2}, 0, 0, 0 \right). \end{aligned} \quad (4.30)$$

This singles out the direction 1 and causes τ to vanish by radiality when one of its indices is equal to one; the other three directions, however, are completely equivalent. The expression in square brackets of (4.29) reduces then to

$$\frac{1}{8} \xi^4 \langle R_{a1\mu 1}(x_0) R_{c1\rho 1}(y_0) \rangle \quad (4.31)$$

and there are three possibilities of choosing the indices $\{a\mu\}$ and $\{c\rho\}$ of the two vierbein fields:

- (a) like in $\langle \tau_{22}\tau_{22} \rangle$ or $\langle \tau_{23}\tau_{23} \rangle$...; this represents the correlation between the same component of the same vector of the vierbein;
- (b) like in $\langle \tau_{22}\tau_{23} \rangle$ or $\langle \tau_{22}\tau_{32} \rangle$...; this is a correlation between different components of the same vector or between the same component of different vectors of the vierbein;
- (c) like in $\langle \tau_{22}\tau_{33} \rangle$ or $\langle \tau_{22}\tau_{44} \rangle$...; that is, a correlation between different components of different vectors, but chosen in such a way that $a = \mu$ and $c = \rho$.

It is straightforward to insert these choices of indices in (4.31) and to evaluate the propagator for each choice. Also, the integrals in λ and t are trivial in this case and the dependence on ξ factorizes as ξ^{-2} , as expected on dimensional grounds. The three cases (a), (b) and (c) give different results, and it is interesting to look at them keeping in mind that each vector of the vierbein is parallel-transported between x_0 and y_0 in any configuration of the functional integral used to produce the averages. It is found the following: The correlation (a), which can also be expressed as $\langle \tau_{ij} \tau_{ij} \rangle$ (no sum), does not vanish. It is in fact the basic correlation function of the radial gauge, as we shall now explain in short. Namely, it will be shown in Section 5.3 that the invariant correlation between two vierbein fields includes the matrix \mathcal{U} of the parallel transport:

$$G_{\text{Vierbein}}(D) = \langle e_\mu^a(x) \mathcal{U}_{aa'}^{\mu\mu'}(x, x') e_{\mu'}^{a'}(x') \rangle$$

(here D is the distance between x and x'). But in radial gauge this matrix reduces to the identity along any ray, thus we have for the choice of coordinates (4.30)

$$\begin{aligned} G_{\text{Vierbein}}(\xi) = & \langle \tau_{22}(x_0) \tau_{22}(y_0) \rangle + \langle \tau_{23}(x_0) \tau_{23}(y_0) \rangle + \langle \tau_{24}(x_0) \tau_{24}(y_0) \rangle + \\ & + \langle \tau_{33}(x_0) \tau_{33}(y_0) \rangle + \langle \tau_{34}(x_0) \tau_{34}(y_0) \rangle + \langle \tau_{44}(x_0) \tau_{44}(y_0) \rangle. \end{aligned} \quad (4.31)$$

This is true to any order. We notice that the terms of (4.31) which differ by the first index of the vierbein are certainly equal, since the vectors of the vierbein are equivalent. Moreover, if no external source is present, different components of the same vector are equivalent too, in the average. One concludes that all the radial correlations of the form $\langle \tau_{ij} \tau_{ij} \rangle$ are equivalent on the vacuum and equal to $\frac{1}{6}$ of the invariant correlation G_{Vierbein} .

The correlation functions (b), also expressible as $\langle \tau_{ij} \tau_{kj} \rangle$ or $\langle \tau_{ij} \tau_{ik} \rangle$ (no sum), vanish to lowest order and are in general not significant, because there is no direct connection (apart from the orthonormality relations) between the values of two different components of the same vierbein vector or the values of the same components of two different vectors.

The correlations (c), also expressible like $\langle \tau_{ii} \tau_{jj} \rangle$ (no sum; $i \neq j$), do not vanish to lowest order. Nevertheless, this property is due to the weak-field approximation and does not hold in general. It is easy to verify by geometrical arguments that on a almost-flat space the i -th component of the i -th vierbein vector keeps constant in sign, so its correlation does not vanish. This ceases to be true on a generic strongly curved space.

5. GAUGE INVARIANT CORRELATIONS.

In the first part of this review (Chapters 2-4) we have shown how some correlation functions of the gravitational field at geodesic distance can be computed in a special physical gauge. By this technique two-point functions are obtained, which really depend on geodesic coordinates x and y (more exactly, they are hypergeometric functions of x and y – see Sections 4.2, 4.3) and have “parallel transported” tensorial components which are a generalization of the usual Lorentz tensorial components (compare Section 3.3).

In this second part (Chapters 5, 6), on the contrary, the most relevant scalar, or “gauge invariant” correlations will be introduced; by this we mean functionals of the field, whose corresponding classical quantities are independent of the choice of the coordinates. The most typical and useful example of this kind of correlations is the Wilson loop of the Christoffel or anholonomic connection (Sections 5.1, 5.4).

The physical interpretation of these quantities turns out to be more clear and interesting than in the case of the gauge-dependent correlations. On the other hand, the “geodesic corrections” are less easy to be taken into account in this case; we shall illustrate a special procedure for that in Section 6.3.

This Chapter is organized as follows. In Section 5.1 the matrix \mathcal{U} of the parallel transport is defined, both in terms of the Christoffel connection $\Gamma_{\mu\nu}^\alpha$ and of the gauge, or “anholonomic” connection $\Gamma_{\mu b}^a$. In Section 5.2 we specify the dynamical scheme as the traditional perturbative scheme which starts from the Einstein lagrangian, splits the gravitational field into a flat “background” and a weak, quantized part. In Section 5.3 the invariant two-point functions involving the Riemann curvature are computed to leading order. They vanish as a consequence of the equations of motion. On the contrary, the correlations of the metric and of the vierbein do not vanish. Finally, in Section 5.4 we compute to leading order the Wilson loop of the Christoffel connection and find that it vanishes too. The physical interpretation of this property is postponed to Chapter 6.

G.1 Geometrical definitions.

We consider a classical spacetime M described by a metric tensor $g_{\mu\nu}(x)$ of signature $(-1, 1, 1, 1)$ (the conventions are those of Weinberg [1972]).

The variation of a vector V^α by an infinitesimal parallel transport is defined by

$$dV^\alpha = -\Gamma_{\mu\beta}^\alpha(x) V^\beta dx^\mu, \quad (5.1)$$

where $\Gamma_{\mu\beta}^\alpha$ is the Christoffel connection

$$\Gamma_{\mu\beta}^\alpha = \frac{1}{2} g^{\alpha\gamma} (\partial_\mu g_{\beta\gamma} + \partial_\beta g_{\mu\gamma} - \partial_\gamma g_{\mu\beta}). \quad (5.2)$$

Integrating (5.1) we find that the parallel transport of V along a finite differentiable curve connecting the points x and x' is performed by the matrix

$$\mathcal{U}_\beta^\alpha(x, x') = \text{P exp} \int_x^{x'} dy^\mu \Gamma_{\mu\beta}^\alpha(y), \quad (5.3)$$

where the symbol P means that the matrices $(\Gamma_\mu)^\alpha_\beta = \Gamma_{\mu\beta}^\alpha$ are ordered along the path. The indices of $\mathcal{U}_\beta^\alpha(x, x')$ are lowered and raised by $g_{\alpha\gamma}(x)$ and $g^{\beta\gamma}(x')$, respectively.

When the manifold is curved, the matrix \mathcal{U} depends not only on the end points x and x' , but also on the path. So, if C is a smooth closed curve on M , we define the loop functional (or “holonomy”) $\mathcal{W}(C)$ as

$$\mathcal{W}(C) = -4 + \text{Tr } \mathcal{U}(C) = -4 + \text{Tr } P \exp \oint_C dx^\mu \Gamma_\mu(x). \quad (5.4)$$

The term -4 sets the holonomy to zero in the case of a flat space, when the matrix \mathcal{U} reduces to an identity matrix.

Under a coordinates transformation $x \rightarrow \zeta$, the matrix \mathcal{U} transforms in the following way

$$\mathcal{U}_\beta^\alpha(x, x') \rightarrow \mathcal{U}_\beta^\alpha(x, x') \left[\frac{\partial \zeta^\gamma}{\partial x^\alpha} \right]_x \left[\frac{\partial x^\beta}{\partial \zeta^\varepsilon} \right]_{x'}. \quad (5.5)$$

For a closed curve, this transformation, being of the form

$$\mathcal{U} \rightarrow \Omega \mathcal{U} \Omega^{-1},$$

does not affect the trace of \mathcal{U} . So the loop $\mathcal{W}(C)$ is invariant with respect to coordinate transformations.

In the first order formalism, the “anholonomic” components of a vector are defined by

$$V^a = V^\mu e_\mu^a(x).$$

The equivalent of (5.1) in terms of the anholonomic connection $\Gamma_{\mu b}^a$ is

$$dV^a = -\Gamma_{\mu b}^a(x) V^b dx^\mu \quad (5.5)$$

and the matrix \mathcal{U} of the finite parallel transport has an expression which is formally the analogue of (5.3), namely

$$\mathcal{U}_b^a(x, x') = P \exp \int_x^{x'} dy^\mu \Gamma_{\mu b}^a(y).$$

We remind (see eq. 3.5) that the relation between the connections $\Gamma_{\mu\beta}^\alpha$ and $\Gamma_{\mu b}^a$ is the following

$$\Gamma_{\mu\beta}^\alpha = e_a^\alpha e_\mu^b \Gamma_{\beta b}^a + e_a^\alpha \partial_\beta e_\mu^a$$

and that the relation between the matrices \mathcal{U}_β^α and \mathcal{U}_b^a is

$$\mathcal{U}_b^a(x, x') = e_\alpha^a(x) \mathcal{U}_\beta^\alpha(x, x') e_b^\beta(x'). \quad (5.6)$$

It is known that gravity in the vierbein formalism has a local Lorentz invariance, since the definition of the vierbein, eq. 3.3, is insensitive to a Lorentz rotation of $e^a(x)$, $e^b(x)$. The connection $\Gamma_{\mu b}^a$ is then completely analogous to an usual gauge connection, and its Wilson loop

$$\mathcal{W}(C) = -4 + \text{Tr}(\mathcal{U}_b^a)(C)$$

is a natural invariant quantity of the theory. But from eq. (5.6) we see that this loop is equal to that defined in (5.4). So the Christoffel connection $\Gamma_{\mu\beta}^\alpha$ and the anholonomic $\Gamma_{\mu b}^a$ connection have the same loop, denoted by $\mathcal{W}(C)$. In the computations we shall employ the connection $\Gamma_{\mu\beta}^\alpha$, which is usually simpler to deal with.

When the exponential in (5.4) is expanded, one obtains terms with 1, 2, 3, ... fields Γ . We introduce the notation, to be used in the following

$$\begin{aligned} \mathcal{U} &= \mathbf{1} + \oint_C dx^\mu \Gamma_\mu(x) + \frac{1}{2} \text{P} \oint dx^\mu \oint dy^\nu \Gamma_\mu(x) \Gamma_\nu(y) + \dots \\ &= \mathbf{1} + \mathcal{U}^{(1)} + \frac{1}{2} \mathcal{U}^{(2)} + \dots \end{aligned} \quad (5.7)$$

and

$$\mathcal{W} = -4 + \text{Tr} \mathcal{U} = \text{Tr} \mathcal{U}^{(1)} + \frac{1}{2} \text{Tr} \mathcal{U}^{(2)} + \dots \quad (5.8)$$

5.2 Dynamics and perturbation scheme.

Let us now introduce dynamics through the Einstein action, which has the form (for any $N > 2$)

$$S = \frac{1}{\kappa^2} \int d^N x \sqrt{g(x)} R(x).$$

A completely consistent quantum theory of this model does not exist yet. In view of applications to lattice gravity, we shall refer in the following, as a possible approximation to the full theory, to the (regularized) functional integral approach (see for instance [Hawking, 1979; Mazur and Mottola, 1990]; compare also Chapter 7).

In the perturbative evaluations of this chapter, however, we follow the “traditional” approach and regard quantum gravity as an ordinary field theory on a fixed flat background. The fundamental field is $\kappa h_{\mu\nu}(x) = g_{\mu\nu}(x) - \delta_{\mu\nu}$, which is subject to the gauge transformations

$$h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \partial_\mu f_\nu(x) + \partial_\nu f_\mu(x), \quad (5.9)$$

where $f_\mu(x)$ is an arbitrary function. Eq. (5.9) represents the action on h of a linearized diffeomorphism $x'^\mu = f^\mu(x)$. It should be noticed, nevertheless, that the argument x remains unchanged in the transformation (5.9): this is what is meant by “fixed background”. The transformation of Γ which corresponds to (5.9) is

$$\Gamma_{\mu\nu}^\alpha(x) \rightarrow \Gamma_{\mu\nu}^\alpha(x) + \partial_\mu \partial_\nu f^\alpha(x). \quad (5.10)$$

Keeping the quadratic part of S and adding to it the harmonic gauge-fixing

$$\frac{1}{2} \left(\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial^\nu h_\mu^\mu \right)^2,$$

we obtain the Feynman-De Witt propagator (compare Section 3.5, where however h differs by a factor κ)

$$\langle h_{\mu\nu}(x) h_{\rho\sigma}(y) \rangle = -\frac{1}{(2\pi)^2} \frac{\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\rho\sigma}}{(x-y)^2}. \quad (5.11)$$

Due to the non-polynomial character of the lagrangian, there are infinitely many interaction vertices; the first two ones, respectively proportional to κ and κ^2 , connect 3 and 4 fields h . Hence the first few orders of perturbation theory are formally very similar to those of Yang-Mills theory [see for instance Veltman, 1976].

As it is known, the Einstein action is not the only action which describes correctly the macroscopic behaviour of gravity. In particular, the $(R + R^2)$ -action (see Chapter 7) has been proposed a long time ago as a renormalizable generalization of General Relativity. Also, in order to make the euclidean Einstein action bounded from below, a “stabilized” euclidean action has been recently proposed. We shall briefly consider the effects of these modified kinds of dynamics on the correlation functions in Section 5.4.

5.3 Two-point functions.

An important point to be clarified about the vacuum correlation functions in gravity is the following: “which are the most suitable field quantities to be correlated”? We recall that the criteria for the existence of gravitational waves [Zakharov, 1973] are usually based on the propagation of the curvature, which is believed to be the most physical effect of gravitation. One is thus led to consider as first candidates the following quantities:

- the Riemann tensor $R_{\beta\mu\nu}^\alpha(x)$;
- the Ricci tensor $R_{\mu\nu}(x) = R_{\mu\alpha\nu}^\alpha(x)$;
- the curvature scalar $R(x) = g^{\mu\nu}(x) R_{\mu\nu}(x)$;
- the “rotation matrix”, or plaquette $\mathcal{R}_\beta^\alpha(x) = R_{\beta\mu\nu}^\alpha(x) \sigma^{\mu\nu}$, where $\sigma^{\mu\nu}$ is an infinitesimal surface around x (\mathcal{R} must not be confused with the “Mandelstam covariant” of Chapter 2).

The linearized parts of R , $R_{\mu\nu}$ and $R_{\beta\mu\nu}^\alpha$ are respectively given by

$$\begin{aligned} R^L &= \partial^2 h_\alpha^\alpha - \partial^\alpha \partial^\beta h_{\alpha\beta}; \\ R_{\mu\nu}^L &= \frac{1}{2} (\partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h_\alpha^\alpha - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu}); \\ R_{\alpha\beta\mu\nu}^L &= \frac{1}{2} (\partial_\alpha \partial_\mu h_{\beta\nu} - \partial_\beta \partial_\mu h_{\alpha\nu} - \partial_\alpha \partial_\nu h_{\beta\mu} + \partial_\beta \partial_\nu h_{\alpha\mu}). \end{aligned}$$

Let us consider the invariant correlations

$$G_R(D) = \langle R(x) R(x') \rangle_0;$$

$$G_{Ricci}(D) = \langle R_{\mu\nu}(x) \mathcal{U}^{\mu\nu\mu'\nu'}(x, x') R_{\mu'\nu'}(x') \rangle_0;$$

$$G_{Riemann}(D) = \langle R_{\beta\mu\nu}^\alpha(x) \mathcal{U}_{\alpha\alpha'}^{\beta\mu\nu\beta'\mu'\nu'}(x, x') R_{\beta'\mu'\nu'}^{\alpha'}(x') \rangle_0;$$

$$G_{Loop}(D, \sigma, \sigma') = \langle \mathcal{R}_\beta^\alpha(x) \mathcal{U}_{\alpha\alpha'}^{\beta\beta'}(x, x') \mathcal{R}_{\beta'}^{\alpha'}(x') \rangle_0.$$

Here \mathcal{U} is the matrix of the parallel transport introduced in Section 5.1, computed along the geodesic joining x to x' ; D is the geodesic distance between x and x' .

On a flat background and to order κ^2 these functions are very simple, since the matrices $\mathcal{U}(x, x')$ reduce to identity matrices and the tensors retain only their linearized parts. It is easy to verify that for the first three correlation functions one has [M., 1992b]

$$G_R \sim G_{Ricci} \sim G_{Riemann} \sim \kappa^2 \partial^2 \delta^4(x - x') + o(\kappa^2). \quad (5.12)$$

Thus these correlation functions vanish if x and x' are distinct, and they are not indicative of the correlation length of the system – at least in this regime of weak fields.

The perturbative result above can be related to a general property of the correlation functions [see also Collins, 1984]. Let us consider a functional integral of the form

$$z = \int d[\phi(x)] e^{-S[\phi(x)]} \quad (5.13)$$

and perform an infinitesimal arbitrary translation $\delta\phi(x)$ of the integration variable $\phi(x)$. Since the integration measure is invariant with respect to such a transformation, we have

$$z = \int d[\phi(x)] e^{-S[\phi(x)]} e^{-\int dx \delta\phi(x) E(\phi(x))},$$

where $E(\phi(x))$ represents the classical equations of motion. Expanding the second exponential and disregarding terms $O(\phi^2)$ we obtain

$$z = \int d[\phi(x)] e^{-S[\phi(x)]} \left(1 - \int dx \delta\phi(x) E(\phi(x)) \right),$$

from which, by comparison with (5.13), we have

$$\int d[\phi(x)] e^{-S[\phi(x)]} \int dx \delta\phi(x) E(\phi(x)) = 0. \quad (5.14)$$

If we choose $\delta\phi(x)$ to be of the form $\varepsilon\delta^4(x)$, we conclude that

$$\langle E(\phi(x)) \rangle_0 = 0;$$

performing in (5.14) one more translation of the form $\varepsilon\delta^4(x')$ we obtain by similar arguments

$$\langle E(\phi(x)) E(\phi(x')) \rangle_0 = 0,$$

and so on. That is, in a quantum field theory the vacuum correlations of field equations vanish.

In the case of Einstein gravity, the equations of motion are

$$R_{\mu\nu}(x) = 0;$$

thus we have to any order

$$\langle R_{\mu\nu}(x) R_{\rho\sigma}(x') \rangle_0 = 0. \quad (5.15)$$

Since in our perturbative approximation for the functions G the indices are contracted, in practice, just by $\delta_{\mu\nu}$ instead of the full metric $g_{\mu\nu}$, from (5.15) follows (5.12).

Going back to the first-order analysis, we notice that the last correlation function, G_{Loop} , is identical, in this approximation, to a Wilson loop computed along a dumbbell-like contour (fig. 2). The evaluation of the Wilson loop will be the subject of the next Section, where we shall see, however, that it vanishes in this approximation. In conclusion, the only gravitational correlation functions which do not vanish to lowest order are those of the metric or of the vierbein

$$G_{Metric}(D) = \langle g_{\mu\nu}(x) \mathcal{U}^{\mu\nu\mu'\nu'}(x, x') g_{\mu'\nu'}(x') \rangle_0;$$

$$G_{Vierbein}(D) = \langle e_\mu^a(x) \mathcal{U}_{aa'}^{\mu\mu'}(x, x') e_{\mu'}^{a'}(x') \rangle_0,$$

as can be easily checked by a computation similar to that of the correlation functions of the curvature. They both behave like (compare also Section 4.4)

$$G(D) \sim \frac{\kappa^2}{D^2}.$$

5.4 Wilson loops.

In the usual gauge theories the Wilson loop of the connection is one of the most important observable quantities ([Kogut, 1979]; see also the Section on the static potential energy, 6.4).

Unfortunately, in quantum gravity the Wilson loop of the Christoffel connection vanishes, to lowest order, along any contour[†]. The proof of this surprising property is straightforward: consider eq.s (5.7), (5.8) and denote by roman letters the corresponding vacuum averages. For instance:

$$\begin{aligned} U &= \langle \mathcal{U} \rangle_0 = \mathbf{1} + \langle \mathcal{U}^{(1)} \rangle_0 + \frac{1}{2} \langle \mathcal{U}^{(2)} \rangle_0 + \dots \\ &= \mathbf{1} + U^{(1)} + \frac{1}{2} U^{(2)} + \dots \\ W &= \langle \mathcal{W} \rangle_0 = -4 + \text{Tr } U \\ &= \text{Tr } U^{(1)} + \frac{1}{2} \text{Tr } U^{(2)} + \dots \\ &= W^{(1)} + \frac{1}{2} W^{(2)} + \dots \end{aligned}$$

The Wilson loop W is given by

$$W = \frac{1}{2} \text{P} \oint dx^\mu \oint dy^\nu \langle \Gamma_{\mu\beta}^\alpha(x) \Gamma_{\nu\alpha}^\beta(y) \rangle + o(\kappa^2), \quad (5.16)$$

where the brackets denote the bare propagator of Γ . Using (5.2), (5.11) we obtain

$$\begin{aligned} \langle \Gamma_{\mu\beta}^\alpha(x) \Gamma_{\nu\alpha}^\beta(y) \rangle &= -\frac{\kappa^2}{(2\pi)^2} \left\{ a_1 \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} + a_2 \delta_{\mu\nu} \partial^2 \right\} \frac{1}{(x-y)^2} = \\ &= -\frac{\kappa^2}{(2\pi)^2} \left\{ a_1 \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \frac{1}{(x-y)^2} + a_2 \delta_{\mu\nu} \delta^4(x-y) \right\}, \quad (5.17) \end{aligned}$$

where a_1 and a_2 are two numerical coefficients. From (5.16), (5.17) one concludes that W vanishes to this order, except for some divergent perimeter contributions due to the δ -function.

In the next chapter we shall illustrate in detail the physical meaning of this vanishing and its consequences. About the causes, it is essentially due to symmetry reasons and to the absence of massive propagating modes. If one writes down the most general form a

[†] Hamber ([1993]; see Chapter 7) argues that also in the non-perturbative sector the Wilson loop “does not have the same interpretation as in gauge theories, since it is not associated with the newtonian potential energy of two static bodies. In ordinary gauge theories at strong coupling the Wilson loop decays like the area of the loop, due to the strong independent fluctuations of the gauge fields at different points in spacetime and ensuing cancellations. In lattice gravity the situation is quite different since the connections cannot be considered as independent variables, and the fluctuations in the deficit angles at different points in spacetime are strongly correlated.” The reasoning we present here holds in perturbation theory and to lowest order. See, however, Section 6.4 for a physical justification of why a closed loop cannot give in gravity the same result it gives in the gauge theories with “charges” of opposite signs.

massless graviton propagator may have while respecting Poincaré invariance[‡], and repeats the calculation of the Wilson loop, one still finds that it vanishes to order \hbar .

For instance, the situation in $(R+R^2)$ -gravity (see Section 7.1) is different. Here R and $R_{\mu\nu}$, unlike in Einstein gravity, can propagate (cfr. Section 5.3). There can be some non-vanishing invariant correlations of the curvature and eq. (5.17) will be replaced by

$$\langle \Gamma_{\mu\beta}^\alpha(x) \Gamma_{\nu\alpha}^\beta(y) \rangle = \left\{ a_1 \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} + a_2 \delta_{\mu\nu} \partial^2 \right\} \left(-\frac{\kappa^2}{(2\pi)^2(x-y)^2} + \mathcal{D}_M(x-y) \right),$$

where \mathcal{D}_M is the propagator of a massive mode. Therefore, the term proportional to a_2 is not ultra-local any more. However, we recall that only the lightest (massless) states with spin two should dominate at large distances [see for ex. Hamber, 1992a].

As it is well known, Einstein's gravity written in the first order formalism is a gauge theory of the Lorentz group (i.e., the action is invariant under local Lorentz transformations), but not of the whole Poincaré group $ISO(3,1)$. A consistent gauge formulation can be obtained only introducing some auxiliary fields q^a [Grignani and Nardelli, 1992].

So it is not possible to consider in (3+1) dimensions, like in (2+1)-gravity [Witten, 1988], the holonomies of the Lie algebra valued connection

$$\mathcal{A}_\mu(x) = e_\mu^a(x) P_a + \Gamma_\mu^{ab}(x) \omega_{ab},$$

where P_a and ω_{ab} are the generators of the translations and of the Lorentz transformations. The holonomies of \mathcal{A}_μ may have more content than the holonomies of Γ_μ alone. For instance, it can be easily verified that the term

$$\text{Tr} \oint dx^\mu \oint dy^\nu \langle e_\mu^a(x) P_a e_\nu^b(y) P_b \rangle_0 = -2\delta_{ab} \oint dx^\mu \oint dy^\nu \langle e_\mu^a(x) e_\nu^b(y) \rangle_0$$

is not trivial to leading order, unlike the corresponding term containing the connection. However, this term does not respect the invariance of the action.

[‡] A non-vanishing contribution proportional to \hbar may arise on a non-flat background [M., 1993 c].

6. FURTHER PROPERTIES OF THE WILSON LOOPS.

6.1 Geometrical and physical interpretation.

As we have seen in Section 5.4, the Wilson loop of the Christoffel connection vanishes to leading order in Einstein's gravity. Also, we remind that this loop is equivalent to that of the gauge connection $\Gamma_{\mu b}^a$ (see eq. (5.6) and the ensuing discussion).

In order to understand the physical meaning of this vanishing, a group-theoretical analysis of the matrix \mathcal{U} is needed. We shall see that in the euclidean theory the vanishing of its trace amounts to a very strong geometrical statement.

Let us first consider, for illustrative purposes, the case of a Yang-Mills theory of the group $SO(3)$. The gauge connection has the form

$$A_\mu(x) = A_\mu^i(x) L_i; \quad i = 1, 2, 3,$$

where the matrices L_i constitute a representation of the Lie algebra of the group. In particular, to fix the ideas, let us choose the adjoint representation; in this case the matrices L_i have elements $(L_i)_n^m$ ($m, n = 1, 2, 3$), which are related to the structure constants ε_{imn} of the group. The connection $A_\mu(x)$ performs the parallel transport of a 3-dimensional vector V^n in the "internal" space according to the formula (compare eq. (5.1))

$$dV^m = A_\mu^i(x) (L_i)_n^m V^n dx^\mu.$$

The vector is rotated during the transport, but its length remains unchanged. Let us consider the matrix $\mathcal{O}(C)$ which describes the parallel transport along a closed curve C . $\mathcal{O}(C)$ is defined by a P-exponential, through a formula similar to eq. (5.3). Suppose that we take a vector V in a point P of C , and parallel-transport it along C , returning to P ; let us denote by V' the new vector we obtain in this way. The vectors V and V' have the same length, that is

$$\delta_{mn} V^m V^n = \delta_{mn} V'^m V'^n, \quad (6.1)$$

but they differ by an angle θ , which is related to the trace of $\mathcal{O}(C)$. For small angles, we have, by a proper choice of the coordinate axes in the internal space

$$\mathcal{O}(C) = \begin{pmatrix} \left(1 - \frac{1}{2}\theta^2\right) & \theta & 0 \\ -\theta & \left(1 - \frac{1}{2}\theta^2\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.2)$$

that means

$$\text{Tr } \mathcal{O}(C) = 3 - \theta^2.$$

More generally, we remind that the Lie algebra of $SO(3)$ has just one Casimir invariant, namely the operator

$$L^2 = L_1^2 + L_2^2 + L_3^2.$$

This operator commutes with each of the L_i 's, so we can in general rotate our coordinate system as to have $L^2 = L_3^2$, and the rotation matrix takes in this case the form (6.2), i.e. we have

$$\mathcal{O}(C) = \mathbf{1} + \theta L_3 + \frac{1}{2}\theta^2 L_3^2 + \dots \quad (6.3)$$

Taking the trace of (6.3), remembering that $\text{Tr } L_i = 0$ and using the normalization condition of the Lie generators

$$\text{Tr } L_i L_j = -2\delta_{ij},$$

we find that θ^2 is the coefficient of the Casimir invariant in the expansion of the exponential.

Next we come to consider the group $SO(4)$. Intuitively, adding a new dimension we can make an independent rotation. Multiplying two 4-dimensional matrices similar to (6.2), the first representing a rotation by an angle θ_I perpendicular to one plane and the second a rotation by another angle θ_{II} perpendicular to another plane, we find that

$$\text{Tr } \mathcal{O}(C) = 4 - (\theta_I^2 + \theta_{II}^2) \quad (6.4)$$

Also we know that $SO(4) = [SO(3)]_I \times [SO(3)]_{II}$ and that we have two Casimirs now [see for instance Wybourne, 1974], corresponding to $(L_I^2 + L_{II}^2)$, whose “eigenvalue” appears in (6.4), and $(L_I^2 - L_{II}^2)$, which is not of interest in this case.

The group $SO(4)$ is the relevant one for euclidean quantum gravity. In fact, the geometrical interpretation of the matrix $\mathcal{U}(C)$ is the following. During the parallel transport of a vector V in spacetime, its length, given by

$$|V|^2 = V^a V^b \delta_{ab} = V^\mu V^\nu g_{\mu\nu}(x),$$

does not change. If we transport V along a closed curve C , returning to the starting point, we obtain another vector V' , which has the same length of V , and differs from it only in the orientation. Hence we have for any vector

$$V^a V^b \delta_{ab} = V'^a V'^b \delta_{ab} = \mathcal{U}_c^a(C) V^c \mathcal{U}_d^b(C) V^d \delta_{ab},$$

or, in matrix notation,

$$\mathcal{U}^T(C) \mathcal{U}(C) = \mathbf{1}.$$

The matrix \mathcal{U} belong then to $SO(4)$ and its trace has the form (6.4).

If the variance of the angles θ_I and θ_{II} is zero to order \hbar (because $W^{(2)}$ vanishes), the angles themselves have to vanish identically in any configuration, that is

$$\mathcal{U}(C) = \mathbf{1} \text{ for any } C.$$

This is a very strong geometrical statement, as it implies that, still to order \hbar , all the weak field configurations which effectively enter the functional integral

$$z = \int d[h] \exp \{-\hbar^{-1} S[h]\} \quad (6.5)$$

have no curvature. In other words, the curved configurations – which possibly dominate in other regimes – are in this approximation totally suppressed.

This unexpected situation should be compared with what happens, for instance, in a ordinary $SO(3)$ or $SO(4)$ gauge theory. In this case the leading term $W^{(2)}(C)$ does not vanish and the variance of the rotation angles is not zero to order \hbar . For instance, if the curve C has the form of a rectangle of sides L and T , with $L \ll T$, the quantity $-(\hbar T)^{-1} \log \langle \theta^2 \rangle_0$ is the potential energy of two non-abelian charges kept at rest at a distance L each from the other.

So the matrices of the parallel transport in the “internal” gauge manifold, considered configuration by configuration, are not equal to the identity matrix. Interpreting \hbar as the temperature Θ of an equivalent statistical system, we see that when Θ grows from zero to some small value – such that we may disregard Θ^2 or higher orders – the Yang-Mills fields develop “localized excitations”, i.e. regions of various sizes where the Yang-Mills curvature is not vanishing.

All this does not happen for the gravitational field, which remains essentially in a “flat” state. Such a picture also explains the absence in this approximation of any invariant correlation of the curvature (compare Section 5.3).

The curved configurations to order \hbar in the functional integral could have been interpreted in a natural way as “virtual gravitons”, since they represent gauge-invariant excitations which are off-shell and localized in space and time. (Virtual “gravitons of the metric” do exist in perturbation theory, but they do not necessarily represent physical objects.) Their absence is a very peculiar property of gravity, which also forces us to imagine a peculiar mechanism for the gravitational interaction (see Section 6.4).

6.2 The dumbbell correlation function.

In this section all the matrix elements of U will be computed to order κ^2 , for a dumbbell-like contour (see fig. 2). To this order we have $U^{(1)} = 0$, while $U^{(2)}$ is given by

$$U^{(2)\alpha}_{\beta} = P \oint_C dx^\mu \oint_C dy^\nu \langle \Gamma_{\mu\gamma}^\alpha(x) \Gamma_{\nu\beta}^\gamma(y) \rangle, \quad (6.6)$$

where C is the dumbbell. $U^{(2)}$ is invariant to order κ^2 , by virtue of (5.10). Although not invariant to higher orders, it constitutes an interesting example of two-point correlation function.

Disregarding gradient terms in the integrand and ultra-local terms of the form $\delta^N(x - y)$, eq. (6.6) becomes, in the Feynman-De Witt gauge and in any dimension $N > 2$

$$\begin{aligned} U^{(2)\alpha}_{\beta} &= \frac{1}{4} \delta^{\gamma\gamma'} \delta^{\alpha\alpha'} P \oint_C dx^\mu \oint_C dy^\nu \langle [\partial_\gamma h_{\alpha'\mu}(x) - \partial_{\alpha'} h_{\mu\gamma}(x)] [\partial_\beta h_{\gamma'\nu}(y) - \partial_{\gamma'} h_{\nu\beta}(y)] \rangle \\ &= \frac{1}{4} c_N \kappa^2 \frac{3N - N^2}{N - 2} P \oint_C dx^\mu \oint_C dy_\mu \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y^\beta} \frac{1}{(x - y)^{N-2}}. \end{aligned} \quad (6.7)$$

It is immediate to verify that the trace of $U^{(2)}$ is an ultra-local term. Eq. (6.7) is remarkable under two aspects. First, it shows that the matrix $U^{(2)}$ is symmetric. Second, we see that it vanishes identically in dimension $N = 3$; this happens because the Riemann curvature in an empty space-time of dimension 3 is always zero (compare the discussion following eq. (5.12)).

When C is a dumbbell the integral breaks down in 16 parts (fig. 3), which may be denoted as follows

$$\begin{aligned} U^{(2)}(r, D) &= u_{AA} + u_{BB} + u_{CC} + u_{DD} + \\ &+ u_{AB} + u_{BA} + u_{AC} + u_{CA} + u_{AD} + u_{DA} + \\ &+ u_{BC} + u_{CB} + u_{BD} + u_{DB} + u_{CD} + u_{DC} = \\ &= u_{AA}(r) + u_{CC}(r) + \\ &+ u_{BB}(d) + u_{DD}(d) + 2u_{BD}(d) + \\ &+ 2u_{AC}(r, D). \end{aligned}$$

In this equation we have put $d = D - 2r$ and omitted the indices α and β . The “transport” terms depending on d can be easily shown to vanish in the limit of zero-width δ of the strip (this holds only for the linearized theory, which behaves like an abelian one).

The most interesting case is that of $r \ll D$. We remind that when a vector is transported around an infinitesimal surface $dx^\mu \wedge dx^\nu$, its variation is given by the matrix

$$M_\gamma^\rho = R_{\gamma\mu\nu}^\rho dx^\mu \wedge dx^\nu,$$

where $R_{\gamma\mu\nu}^\rho$ is the Riemann curvature. Like in Section 5.3, let us denote by \mathcal{R}_γ^ρ the matrix M_γ^ρ divided by the area, in the case of the surface being a small geodesic circle. \mathcal{R}_γ^ρ is a kind of “regularized curvature”; the regularization is non-local and gauge invariant, at the linearized level. Thus the integral $U^{(2)\alpha}_\beta(r, D)$ may symbolically be written in this limit as

$$U^{(2)\alpha}_\beta(r, D) = 2\pi^2 r^4 \left\{ \langle \mathcal{R}_\gamma^\alpha(0) \mathcal{R}_\beta^\gamma(0) \rangle + \langle \mathcal{R}_\gamma^\alpha(0) \mathcal{R}_\beta^\gamma(D) \rangle \right\}.$$

The first term on the r.h.s. is divergent but independent on D and the second one is the “dumbbell function” (fig. 4)

$$\langle \mathcal{R}_\gamma^\alpha(0) \mathcal{R}_\beta^\gamma(D) \rangle = \lim_{a \rightarrow 0} \frac{-c_N \kappa^2 (3N - N^2)}{4\pi^2 a^2 D^{N+2}} \int_0^{2\pi} ds \int_0^{2\pi} dt \frac{1}{2} \cos(s-t) k^N \phi_\beta^\alpha(s, t, a),$$

where $= r/D$ and the non-zero matrix elements are

$$\begin{aligned} \phi_1^1(s, t, a) &= 1 - Nk^2(1 - au)^2; \\ \phi_2^2(s, t, a) &= 1 - Nk^2 a^2 v^2; \\ \phi_2^1(s, t, a) &= \phi_1^2(s, t, a) = -N k^2 a v (1 - au); \\ \phi_\mu^\mu(s, t, a) &= 1; \quad \mu = 3, 4, \dots \end{aligned}$$

with

$$k = 1/\sqrt{1 - 2au + 2a^2w}; \\ u = \cos s - \cos t; \quad v = \sin s - \sin t; \quad w = 1 - \cos(s - t).$$

Expanding these elements in powers of a and integrating[†], one finds that the first non-zero coefficients are those of a^2 . In this way, denoting by Φ the integral of ϕ with respect to s, t , one finally obtains

$$\langle \mathcal{R}_\gamma^\alpha(0) \mathcal{R}_\beta^\gamma(D) \rangle = \frac{-c_N \kappa^2 (3N - N^2)}{4\pi^2 D^{N+2}} \Phi_\beta^\alpha; \\ \Phi_1^1 = \frac{1}{2} N(1 + N)(N - 2); \\ \Phi_2^2 = -\frac{1}{2} N(N - 2); \\ \Phi_2^1 = \Phi_1^2 = 0; \\ \Phi_\mu^\mu = -\frac{1}{2} N^2, \quad \mu = 3, 4, \dots \quad (6.8)$$

In eq. (6.8) one can verify one more time that the trace vanishes. Actually, this is the point where this fact was noticed for the first time.

6.3 Geodesic corrections.

This section will be concerned with a problem raised by the presence of a “dynamic metric” in quantum gravity. The idea of a dynamic geometry is inherent to General Relativity, but it could not be applied to the quantized theory as long as it was considered as a field theory on a fixed background. A recent realization of this idea can be found in the Montecarlo simulations of Regge calculus (see Chapter 7).

Our attention still focuses on the loops of the connection. A natural question arises: how can they be defined in the absence of a flat background?

Let us first consider the case of a classical gravitational field. We assume spacetime to be described by a manifold M , endowed with a metric g . Let us consider on M a closed smooth curve C ; C must be defined in an intrinsic fashion, without reference to any coordinate system. In particular, it is important to specify the *form* and the *size* of C , as in the two following examples:

(1) the circle of geodesic radius r (fig. 5). It is defined as follows: given a center point O and a plane through O , we go along the geodesic lines which start from O tangent to the plane and we stop when the invariant distance from O is r . The points we reach in this way belong to the circle.

[†] This is a quite long calculation, which can be easily performed using for instance the symbolic program *Mathematica*.

(2) the (planar) “dumbbell” (fig. 6). It consists of two circles of geodesic radius r , whose centers are placed along a geodesic line, the second lying an invariant distance D apart from the first. The circles are joined by a strip of small width δ ($\delta \rightarrow 0$).

In a curved space with arbitrary metric the geodesic circles and the geodesic dumbbells may “look” quite different from those in the flat space. When we compute the vacuum average of a quantity like W through a numerical simulation of Regge calculus, what happens, intuitively, is the following: the computer produces an arbitrary field configuration, with a probability weighted by the exponential of the euclidean action; on this configuration W is “measured”, along a contour defined in a similar way as we did for the geodesic circle or the geodesic dumbbell; these steps are repeated for many configurations and finally the average of the results is computed.

On the other hand, the attitude one takes in the perturbative calculations, like in Chapter 5, is different. One works on a flat background, looking at small fluctuations of the metric around the background, but disregarding the effect of these fluctuations on the definition of the contours.

It is possible to reconcile the two views in the case of a weak field, by introducing “geodesic corrections” in the perturbative calculations on a flat background [M., 1993a]. We start recalling some useful properties of the normal coordinates $\{\xi^\mu\}$. Chosen an origin O , they can be defined in a neighbourhood of O imposing the condition

$$\xi^\mu \xi^\nu \Gamma_{\mu\nu}^\rho(\xi) = 0. \quad (6.9)$$

As a consequence of (6.9), the lines of the form $\xi^\mu(\lambda) = \lambda v^\mu$, where v is a fixed vector, represent geodesic lines through O . Furthermore, the invariant distance between a point ξ and the origin is simply given by $\sqrt{\xi^\mu \xi^\nu \delta_{\mu\nu}}$. It follows that a geodesic circle of radius r and center O has the familiar form

$$\xi(s) = (r \cos s, r \sin s, 0, \dots); \quad 0 \leq s \leq 2\pi. \quad (6.10)$$

Given any coordinate system $\{x^\mu\}$ with origin O , the normal coordinates ξ can be computed from the coordinates x by solving the system (3.11), (3.12). We are interested in the first-order perturbative solution given by (3.13), namely

$$x^\mu(\xi) = \xi^\mu + X^\mu(\xi) + O(\Gamma^2), \quad (6.11)$$

where

$$X^\mu(\xi) = -\xi^\rho \xi^\sigma \int_0^1 dt (1-t) \Gamma_{\rho\sigma}^\mu(t\xi), \quad (6.12)$$

being Γ the connection in the coordinates x .

The corrections due to (6.11) involve in the term $\mathcal{U}^{(1)}$, which on a flat background gives no contributions to order κ^2 . We choose $\{x^\mu\}$ to be the harmonic coordinates fixed by the Feynman-De Witt gauge. In order to fix the ideas we also make the hypothesis that C is a geodesic circle, although this hypothesis is in fact irrelevant. The function $x^\mu(s)$

which represents the circle in the coordinates x can be obtained from eq.s (6.10), (6.12). Since we are interested in terms up to order Γ^2 (corresponding to order κ^2 in the quantum perturbative series), the approximation (6.12) will be sufficient. We thus have

$$\begin{aligned} \mathcal{U}^{(1)\alpha}_\beta &= \oint_C dx^\mu(\xi) \Gamma_{\mu\beta}^\alpha[x(\xi)] = \\ &= \oint_C d\xi^\mu \Gamma_{\mu\beta}^\alpha(\xi) + \oint_C dX^\mu(\xi) \Gamma_{\mu\beta}^\alpha(\xi) + \\ &+ \oint_C d\xi^\mu \partial_\nu \Gamma_{\mu\beta}^\alpha(\xi) X^\nu(\xi) + O(\Gamma^3), \end{aligned} \quad (6.13)$$

where ξ is given by (6.10). In eq. (6.13), the first integral represents the naive contribution, which vanishes when it is averaged on the vacuum; the second integral is a kind of “index” correction; the third one is a correction to the “argument”. Using (6.12), one finds for the average value of $\mathcal{U}^{(1)}$ on the vacuum state

$$U^{(1)\alpha}_\beta = - \int_0^1 dt (1-t) f_\beta^\alpha(t) + o(\kappa^2), \quad (6.14)$$

where

$$\begin{aligned} f_\beta^\alpha(t) &= \oint_C d(\xi^\rho \xi^\sigma) \langle \Gamma_{\mu\beta}^\alpha(\xi) \Gamma_{\rho\sigma}^\mu(t\xi) \rangle + \\ &+ t \oint_C d\xi^\nu \xi^\rho \xi^\sigma \langle \Gamma_{\mu\beta}^\alpha(\xi) \partial_\nu \Gamma_{\rho\sigma}^\mu(t\xi) \rangle + \\ &+ \oint_C d\xi^\mu \xi^\rho \xi^\sigma \langle \partial_\nu \Gamma_{\mu\beta}^\alpha(\xi) \Gamma_{\rho\sigma}^\nu(t\xi) \rangle. \end{aligned}$$

Disregarding gradient terms in the integrand we obtain

$$f_\beta^\alpha(t) = \delta^{\alpha\alpha'} \delta_{\gamma\tau}^{\nu\mu} \oint_C d\xi^\tau \xi^\rho \xi^\sigma \left\langle [\partial_\beta \partial_\nu h_{\alpha'\mu}(\xi)]_{[\alpha'\beta]} \Gamma_{\rho\sigma}^\gamma(t\xi) \right\rangle. \quad (6.15)$$

The antisymmetry of f_β^α causes its trace to vanish. It follows that the geodesic contribution of order κ^2 to W vanishes, like the naive contribution $\text{Tr } U^{(2)}$. We notice that the antisymmetry of f in (6.15) does not depend on the form of the propagator, and is thus a geometric property. On the contrary, we remind that the trace of the naive contribution is sensitive to the dynamics of the field (compare Section 5.4). We finally point out that eq. (6.15) holds for any closed curve C , not just for the geodesic circle.

Returning to (6.15), let us now compute all the matrix elements. We notice that eq. (6.15), substituted into (6.14), produces a correction of the form

$$U^{(1)} = \int_0^1 dt (1-t) \oint_C d\xi \langle \Psi(\xi) \Psi(t\xi) \rangle, \quad (6.16)$$

where Ψ is a field. This expression shows that the geodesic contribution is related to the necessity of fixing a scale for the distances. Since (6.16) diverges for $t \sim 1$, in the following we shall regularize it by stopping the integrations at $t = 1 - \epsilon$.

Substituting the propagator (5.11) into (6.15) we find, after some steps, in any dimension $N > 2$

$$U^{(1)\alpha}_{\beta} = -c_N \kappa^2 \delta^{\alpha\alpha'} \int_0^{1-\epsilon} \frac{dt}{(1-t)^N} \oint_C d\xi^\tau \xi^\rho \xi^\sigma \times \\ \times \left\{ \left(\frac{3N - N^2}{N-2} \right) \delta_{\alpha'\rho} \partial_\beta \partial_\tau \partial_\sigma \frac{1}{\xi^{N-2}} - \left(\frac{2}{N-2} \right) \delta_{\alpha'\tau} \partial_\beta \partial_\rho \partial_\sigma \frac{1}{\xi^{N-2}} \right\}_{[\alpha'\beta], (\rho\sigma)}.$$

Finally, disregarding gradient terms in the integrand we obtain

$$U^{(1)\alpha}_{\beta} = -c_N \kappa^2 \chi_N^{(1)}(\epsilon) \oint_C \frac{1}{2} [d\xi^\alpha \partial_\beta - d\xi_\beta \partial^\alpha] \frac{1}{\xi^{N-2}} = \\ = -c_N \kappa^2 \chi_N^{(2)}(\epsilon) \oint_C d(\xi^\alpha \wedge \xi_\beta) \frac{1}{\xi^N}, \quad (6.17)$$

where $\chi_N^{(i)}(\epsilon)$ diverges like ϵ^{1-N} when $\epsilon \rightarrow 0$. For a geodesic circle of radius r in the plane 1-2, the only non zero elements of $U^{(1)}$ are simply

$$U^{(1)1}_2 = -U^{(1)2}_1 = -c_N \kappa^2 \chi_N^{(2)}(\epsilon) \frac{\pi}{r^{N-2}}.$$

For a geodesic dumbbell we have, in the limit $r \ll D$,

$$U^{(1)1}_2 = -U^{(1)2}_1 \simeq -c_N \kappa^2 \chi_N^{(2)}(\epsilon) \frac{\pi r^2}{\left(\frac{D}{2}\right)^N}.$$

This correction should be added to (6.7). Although it is formally quite clear, a further analysis seems to be necessary in order to understand its physical meaning.

We also remind that the matrix (6.17) is not invariant under gauge transformations. Nevertheless, its simplicity suggests us that it could have a quite general geometrical meaning.

6.4 The static potential energy.

It is known that the static potential energy $U(L)$ of two sources of a gauge field is related to the Wilson loop of temporal size T and spatial size $L \ll T$ by the formula

$$e^{-\hbar^{-1} T U(L)} = W(L, T).$$

In the strong coupling limit this expression leads to the confining potential $U(L) = kL$, while in the weak coupling limit one easily recovers the Coulomb potential $U(L) = -e^2/L$.

The vanishing of W to leading order in gravity raises the problem of finding another invariant expectation value of the quantized field which gives the static potential energy between two masses. This problem has a well defined solution indeed [M., 1993b]. One starts from the following known formula of Euclidean field theory

$$\mathcal{E} = \lim_{T \rightarrow \infty} -\frac{1}{\hbar T} \log \frac{\int d[\phi] \exp\{-\hbar^{-1}(S_0[\phi] + S_{\text{Inter.}}[\phi, J])\}}{\int d[\phi] \exp\{-\hbar^{-1}S_0[\phi]\}}, \quad (6.18)$$

where ϕ is a quantum field and J is a classical source coupled to ϕ , which is switched off outside the interval $(-\frac{1}{2}T, \frac{1}{2}T)$. \mathcal{E} represents the energy of the ground state of the system. We shall only show here how this formula works in the case of a weak gravitational field on a flat background. Replacing ϕ with the gravitational field, S_0 with Einstein's action and J with two particles of masses m_1, m_2 , following the trajectories

$$x^\mu(t_1) = \left(t_1, -\frac{L}{2}, 0, 0 \right); \quad y^\mu(t_2) = \left(t_2, \frac{L}{2}, 0, 0 \right), \quad (6.19)$$

we find (for a weak field)

$$\mathcal{E} = \lim_{T \rightarrow \infty} -\frac{1}{\hbar T} \times \times \log \frac{\int d[h] \exp \hbar^{-1} \left\{ -S_{\text{Einst.}}[h] - m_1 \int dt_1 \sqrt{1 - h_{00}[x(t_1)]} - m_2 \int dt_2 \sqrt{1 - h_{00}[y(t_2)]} \right\}}{\int d[h] \exp\{-\hbar^{-1}S_{\text{Einst.}}[h]\}}.$$

It is easy to verify that to leading perturbative order this gives the correct result

$$\mathcal{E} = m_1 + m_2 - \frac{m_1 m_2 G}{L}.$$

It is interesting to make a comparison with electrodynamics. In that case the analogue of the functional integral which appears in the logarithm of (6.18) has the form

$$\left\langle \exp \left\{ e \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 A_0[x(t_1)] - e \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 A_0[y(t_2)] \right\} \right\rangle. \quad (6.20)$$

(The two charges have been chosen to be opposite: $q_1 = e, q_2 = -e$.) Reversing the direction of integration in the second integral and closing the contour at infinity, one is able to show that the quantity (6.21) coincides with the Wilson loop of a single charge g , thus giving a gauge invariant expression for the potential energy.

In gravity this is not possible: we may imagine that an expression like (6.21) could be obtained in the first-order formalism (with A_0 replaced by the tetrad e_0^0), but the masses necessarily have the same sign, so the loop cannot be closed.

It is possible to give a definite meaning to the expression (6.18) for the energy also in the case when no background is present. It can also be proven that the gravitational interaction energy is always negative [M., 1993b]. A very recent work based on this formula is that of Hamber and Williams [1994].

7. LATTICE GRAVITY.

For quantum field theories with relevant non-perturbative effects, the lattice discretization is often quite important, since it allows numerical evaluations through the Montecarlo simulation technique. This is also the case for quantum gravity, which is usually thought to be very strong at small distances, where the theory should somehow determine the small scale “quantized” structure of spacetime [see Rovelli, 1991 and ref.; Smolin, 1992 and ref.].

In the discretized theory the correlation length is a crucial quantity. For this reason we shall briefly recall here some advances made in lattice gravity during the last years, especially from Hamber, Caracciolo and Pelissetto, Ambjørn and co-workers and Greensite.

The point of view of the lattice [see for instance Kogut and Wilson, 1974; Kogut, 1979; Seiler, 1982; Creutz, 1983] is very different from that of continuum field theory; the two are connected, however, by the “continuum limit”.

Let us consider the equivalent statistical system of a field theory discretized on a euclidean lattice. We remind that if the transition between two phases of the system is of the second order, then the appearance of long range correlations causes the details of the lattice to become irrelevant. The original (renormalizable) continuum theory can thus be reobtained, at the transition point, when the lattice spacing goes to zero. If the original continuum theory is not renormalizable in the usual sense (like Einstein gravity), a more general procedure, called “asymptotic safety” [Weinberg, 1979] can be possibly applied.

The chapter is organized as follows. As schematically shown in Table 1, the study of a field theory on the lattice starts choosing one specific version of the theory and a discretization scheme. Some possibilities for gravity are displayed in the table for illustration purposes, with no attempt to completeness; in fact, we are not interested here to make a review of all the different approaches and to compare the results (which should be the same, at least for physical quantities!); we are interested here to the vacuum correlations at geodesic distance. So we have chosen to analyze in more details the way which seems to be the most promising under this respect, namely: $(R + R^2)$ -gravity (Section 7.1), discretized through Regge calculus (Section 7.2), as it appears in the simulations of Hamber (Section 7.3). The remaining section, 7.4, is dedicated to a brief description of other approaches.

Throughout the chapter we shall often quote from the review of Menotti [1990], were many omitted details can be found.

7.1 $(R + R^2)$ -gravity.

It is known that the scalar curvature R is the only scalar which contains second derivatives of the metric. Further scalars can be constructed, however, which contain higher derivatives, namely R^2 , $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ (among the three, only two are independent). The Einstein’s action can thus be modified as follows:

$$S = - \int d^4x \sqrt{g} \left[\frac{1}{\kappa^2} R - \frac{a}{4} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - \frac{1}{3} \left(b - \frac{a}{4} \right) R^2 \right]. \quad (7.1)$$

The terms quadratic in the curvature would not be observable in macroscopic phenomena, but they are important to the quantum theory, since the first radiative corrections to

Einstein's lagrangian contain just R^2 terms. As Stelle showed in detail [1977], $(R + R^2)$ -gravity can be considered as a renormalizable, although non unitary, theory†. Various attempts of recovering unitarity by graphs resummation, suitable gauge fixing and other techniques [Antoniadis et al., 1986; see also Weinberg, 1979] did not lead to any definitive outcome.

The authors of numerical simulations on $(R + R^2)$ models believe usually that the loss of unitarity is a problem confined to perturbation theory, as the whole theory could still be unitary [Hamber, 1986].

7.1 Regge calculus.

The basic idea of this formalism [Regge, 1961] is to replace the smooth spacetime manifold with a simplicial manifold, i.e., a manifold made up of flat regions ("simplexes") connected by "edges". This is easy to figure out in two dimensions (see fig. 7). The discontinuity in the parallel transport arises when an edge is crossed, and the curvature concentrates in those points where a few edges meet, called "hinges". If we transport a vector along a closed path, we only notice a difference between the original vector and the transported one, if the path contains at least one hinge. In a generic dimension N the dimension of the hinges is $N - 2$; hence for $N = 2$ they are points, but for $N = 3$ they are lines and for $N = 4$ they are triangles.

Given an hinge and one simplex which touches it, the opening angle θ_N of the simplex with respect to the hinge can be defined. In two dimensions, the geometrical meaning of such an angle is apparent. In general, θ_N is given by the expression

$$\sin \theta_N = \frac{N}{N-1} \frac{V_N V_{N-2}}{V_{N-1} V'_{N-1}},$$

where V_N is the volume of the mentioned N -dimensional simplex and V_{N-1} , V'_{N-1} are the volumes of the two $(N-1)$ -dimensional simplexes which share the hinge (in the two-dimensional case, they are the two edges which meet at the hinge).

Let us now consider all the simplexes which meet at a given hinge and sum their deficit angles referring to that hinge. We obtain in this way a "deficit angle"

$$\delta_h = 2\pi - \sum_{s \subset h} \theta_s.$$

Regge postulated the following, simple action:

$$S = 2 \sum_h A_h \delta_h, \tag{7.2}$$

where the sum ranges over all hinges and A_h is the surface of the hinge. In the continuum limit, the action (iliz) really approaches Einstein's action (for references about the

† It has been shown that in a suitable range of the couplings a and b the theory is asymptotically free with respect to them. Also, asymptotic freedom with respect to G may arise under certain circumstances [Julve and Tonin, 1978]

exact meaning of this limit and the uniqueness of Regge's action, see [Menotti, 1990] and [Williams and Tuckey, 1991]).

The extension of Regge's action by a cosmological term of the form $\lambda \sum_s V_s$ is straightforward. On the contrary, the lattice version of the R^2 terms is quite involved [for a review see Hamber, 1986]. To write them, one must associate to each hinge a suitably defined volume V_h . The simplest R^2 term is

$$\sum_h \frac{A_h^2 \delta_h^2}{V_h} \rightarrow \frac{1}{4} \int d^N x \sqrt{g(x)} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}.$$

The other terms are more complex, as they entail the introduction of a lattice analogue of the Riemann tensor. All the lattice translations are not unique; however, they are equivalent in the continuum limit – when the maximum length of the edges vanishes, with respect to the characteristic radius of curvature of the smooth manifold.

7.3 The simulations of Hamber.

In a remarkable series of papers starting in 1985, Hamber has developed a technique for simulating quantum Regge calculus of $(R + R^2)$ -gravity [see Hamber, 1986, 1992, 1993 and references therein].

The typical quantities which were first computed in these simulations were rather reminiscent of Ising model, namely the average curvature \bar{R} and the average curvature squared \bar{R}^2 . It turned out that they diverge along a line which goes across the parameters space (a, κ) , thus dividing it into two phases, called “smooth phase” and “rough phase”. The critical indices were computed too.

Recently, the evaluation of two-point correlation functions at geodesic distance has been addressed [Hamber, 1993 b]; the results are interesting although rather preliminary.

In a typical simulation, the simplicial lattice is built up of rigid hypercubes, which can be subdivided into simplices by introducing face diagonals, body diagonals and hyperbody diagonals. This choice is not unique and is dictated by a criterion of simplicity, with the advantage that such a lattice can be used to study rather large systems with little modification. The length of the edges l is individually varied (by moving at random through the lattice), and a new trial length is accepted with probability $\min(1, e^{-\Delta S})$, where ΔS is the variation of the action under the change in edge length. If the triangle inequalities or their higher-dimensional analogues are violated, the new edge length is rejected. It should be noticed that in order to compute the variation in the action under the change of one edge length, a large number of adjoining triangles and their deficit angles has to be considered.

Lattices of size between 4^4 (with 3840 edges) and 16^4 (with 983040 edges) were considered. Periodic boundary conditions (four-torus) were used, since it is expected that for this choice boundary effects should be minimized. The lengths of the runs typically varied between 10-40k Montecarlo iterations on the 4^4 lattice, 2-6k on the 8^4 lattice and 0.5k on the 16^4 lattice. On the larger lattices duplicated copies of the smaller lattices are used as starting configurations, allowing for additional equilibration sweeps after duplicating the lattice in all four directions.

Like we said above, typical quantities which are “measured” in the simulations are the average curvature

$$\bar{R} = \langle l^2 \rangle \frac{\langle 2 \sum_h \delta_h A_h \rangle}{\langle \sum_h V_h \rangle} \sim \frac{\langle \int \sqrt{g} R \rangle}{\langle \int \sqrt{g} \rangle}$$

and the average of the curvature squared

$$\bar{R}^2 = \langle l^2 \rangle^2 \frac{\langle 4 \sum_h \frac{\delta_h^2 A_h}{V_h} \rangle}{\langle \sum_h V_h \rangle} \sim \frac{\langle \int \sqrt{g} R_{\mu\nu\rho\sigma}^2 \rangle}{\langle \int \sqrt{g} \rangle}.$$

One can also estimate the local fluctuation

$$\chi_{\bar{R}} = \frac{1}{\langle \sum_h V_h \rangle} \left[\langle \left(2 \sum_h \delta_h A_h \right)^2 \rangle - \langle 2 \sum_h \delta_h A_h \rangle^2 \right].$$

A divergence in χ should be indicative of a second-order phase transition (as it happens for the magnetic susceptibility or the specific heat).

As the bare Newton constant and the coupling a are varied (compare eq. (7.1), where one sets in this case $b = a/4$), a continuous phase transition is found, separating a “smooth” phase from a “rough” phase. In the first phase the curvature is small and negative, and the fractal dimension is consistent with four. In the second phase the simplices are collapsed, the curvature is large and positive, and the fractal dimension is much less than four, indicating the presence of finger-like structures. Approaching the critical point from the only physically acceptable phase, the smooth one, it was found that the curvature vanishes with an exponent $\delta = 0.62(5)$. At the critical point the curvature fluctuation χ diverges, leading to the possibility of defining a non-trivial lattice continuum (this can be then checked measuring the correlations: see below).

We notice thus that the vacuum expectation value of the curvature can be used as an order parameter for the transition; moreover, it can be used as a possible definition of the effective, long-distance cosmological constant. Usually one adds to the action (7.1) a cosmological term with some bare λ ; the effective cosmological constant is then given by

$$\left(\frac{4\lambda}{\kappa} \right)_{\text{eff}} = \frac{\langle \int \sqrt{g} R \rangle}{\langle \int \sqrt{g} \rangle}.$$

As one approaches the fixed point at κ_c (with fixed a), one finds as expected $(\lambda/\kappa)_{\text{eff}} \rightarrow 0$.

The computation of the correlations at geodesic distance has been recently addressed, in particular for the averages

$$G_R(D) \sim \langle \sum_{h(x)} \delta_h A_h \sum_{h'(y)} \delta_{h'} A_{h'} \delta(|x - y| - D) \rangle,$$

which corresponds to the correlation of the scalar curvature, namely

$$\sim \langle \sqrt{g(x)} R(x) \sqrt{g(y)} R(y) \delta(|x - y| - D) \rangle$$

and

$$G_V(D) \sim \left\langle \sum_{h(x)} V_h \sum_{h'(y)} V_{h'} \delta(|x - y| - D) \right\rangle,$$

which corresponds to the volume correlations

$$\sim \langle \sqrt{g(x)} \sqrt{g(y)} \delta(|x - y| - D) \rangle.$$

The computation proceeds as follows. First, the geodesic distance between any two points x and y is determined in a fixed background geometry. Next the correlations are computed for all pairs of points within geodesic distance D and $D + \Delta D$, where ΔD is an interval slightly larger than the average lattice spacing $l_0 = \sqrt{\langle l^2 \rangle}$, but much smaller than the distance D considered. Finally, the correlations determined for a fixed geodesic distance D are averaged over all the metric configurations.

In principle one could also compute correlations of vector and tensor quantities introducing the matrix of the parallel transport (compare Chapter 5), but this is quite complicated on a Regge lattice and has not be done yet.

The details of the method are described in the mentioned paper. The main result is that the volume correlations are negative at large distances, while the curvature correlations are always positive. If the correlations are fitted to an exponential decay, one finds that the behavior is always consistent with a mass that decreases when one approaches the critical point. This behavior is precisely what is expected if the model is supposed to reproduce the classical Einstein theory for distances which are very large compared to the ultraviolet cutoff scale. Note that this happens for a model of gravity which at short distances is far removed from the pure Einstein theory, containing both a bare cosmological term and bare higher derivative lattice terms!

7.4 The gauge approach and others.

We shall describe the main features of the gauge formalism in dimension $N = 4$, referring to the literature for issues like the reflection positivity, the invariance under reparametrization transformations and the phenomenon of graviton doubling (see Menotti [1990] and references; Smolin [1979]; Menotti and Pelissetto [1986, 1987]).

In the gauge approach one exploits the (incomplete) analogy between gravity and the gauge theories. The attitude is taken of considering gravity as a field theory in Minkowski or euclidean space, whose lagrangian is invariant under a group of local transformations (reparametrization transformations). Several people have introduced along these lines discretized versions of gravity. We shall briefly describe here the formulation due to Smolin, which can be considered as the prototype of these gauge formulations. Essentially Smolin's formulation is the discretized version of the Mac Dowell – Mansouri gauge formulation of De Sitter gravity [1977]. Let us consider the De Sitter group $O(4, 1)$, which goes to $O(5)$ in the euclidean, and introduce the usual hypercubic lattice, familiar to gauge theories. Associate now to each link of the hypercubic lattice a finite element of $O(5)$

$$U(n, n + \mu) = \exp \left(\frac{a}{2} J_{bc} \omega_\mu^{bc} + a P_c e_\mu^c \right),$$

where J_{ab} and P_a satisfy the following commutation relations

$$\begin{aligned} [J_{ab}, J_{cd}] &= \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc} \\ [J_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b \\ [P_a, P_b] &= -J_{ab}. \end{aligned}$$

The action takes the form

$$S = \alpha \sum \epsilon^{\mu\nu\lambda\rho} U_{\mu\nu}^{AB}(n) U_{\lambda\rho}^{CD}(n) \epsilon_{ABCD5},$$

where $\alpha = \frac{3}{8\lambda\kappa^2}$ and

$$\begin{aligned} U_{\mu\nu}^{AB}(n) &= U^{AC}(n, n + \mu) U^{CD}(n + \mu, n + \mu + \nu) \times \\ &\quad U^{DE}(n + \mu + \nu, n + \nu) U^{EB}(n + \nu, n) \end{aligned}$$

is the plaquette in the vector representation. It is clearly invariant under local $O(4)$ -transformations, but not under $O(5)$. This is a consequence of the incomplete similarity between gravity and gauge theories. In the formal continuum limit we obtain the Mac Dowell-Mansouri action

$$S = - \int d^4x e \left[\left(\frac{1}{\kappa} R - \lambda \right) - \frac{3}{8\lambda\kappa^2} \epsilon^{\mu\nu\lambda\rho} R_{\mu\nu}^{ab} R_{\lambda\rho}^{cd} \epsilon^{abcd} \right],$$

which is the familiar Einstein action, plus a cosmological and a Gauss-Bonnet term. The discretized action is reparametrization invariant only in the formal continuum limit.

The extension of the procedure above to the Poincaré group has been given by Menotti and Pelissetto [1987]. To this end one has to introduce in the formalism the local tetrads, which transform locally under $O(4)$. They are defined by the expression

$$E_\mu^a(n) = -\frac{1}{4a} \text{tr} [H P^a U(n, n + \mu) H U(n + \mu, n)],$$

where $H = \text{diag}(1, 1, 1, 1, -1)$.

The gauge approach is strictly bounded to the program outlined by Weinberg of finding a fixed point around which the theory could possibly be renormalized.

The simulations were performed by Caracciolo and Pelissetto [1987, 1988; see also Menotti, 1990] on the De Sitter model in the gauge formulation. They made extensive simulations on an 8^4 lattice. Due to the compact nature of the group $SO(5)$ and the presence of the lattice cut-off, the action is limited and thus the theory must exhibit a well defined ground state. For small values of α the vacuum is similar to the QCD vacuum in the strong coupling region. The large α region is characterized by the vanishing of the vierbein, so that the action becomes dominated by the topological term. An interesting order parameter is given by

$$P = \frac{R_{\mu\nu}^{ab} \tilde{R}_{\mu\nu}^{ab}}{R_{\mu\nu}^{ab} R_{\mu\nu}^{ab}},$$

whose distribution is peaked around zero in the small α region and around the extremes ± 1 in the large α region. At the value $\frac{\alpha}{4} = 0.08 \pm 0.01$ a sudden phase transition is observed. Around this value all measured quantities, like the mean value of the action per site, the trace of the metric, the $O(4)$ curvature and P , exhibit very large hysteresis cycles, thus showing that the transition is first order. For $\alpha > \alpha_c$ the system ends up in non reproducible states showing the existence of many metastable states above α_c and recalling a situation similar to a spin glass.

These results are very different, also qualitatively, from those of Hamber. Unfortunately, such disagreements are not unusual in the numerical simulations of very complex models. In a third approach [Ambjørn et al., 1992] a transition is found, that could be first- or second-order; furthermore, the average curvature at the transition is non zero, which makes the continuum limit quite difficult to understand.

Finally, it is worth mentioning that recently Greensite has given a new interesting approach to euclidean quantum gravity [Greensite, 1993 and references]. It is known that the euclidean Einstein action is not bounded from below, and this means that the euclidean functional integral is not well defined. There have been various attempts to solve this problem [see for instance Mazur and Mottola, 1990 and references]. The method of Greensite is in practice a prescription which can be related to stochastic quantization and which prevents the field configurations in the functional integral from running into the singular configurations which make the action unbounded. This method leads to an effective action which reproduces Einstein's equations but is non-local beyond first order. Although higher order calculations are difficult, it can be proved in a nice way [Greensite, 1992] that the effective cosmological constant generated by this model is vanishing. The Montecarlo simulations are still at an initial stage.

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